The Kumaraswamy Marshal-Olkin Flexible Weibull Distribution and Its applications to Complete Sample

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ABSTRACT

Based on the Kumaraswamy Marshal-Olkin distribution (Gauss M. Cordeiro et al 2015), we study the so-called the Kumaraswamy Marshal-Olkin Flexible Weibull ("KUMOFW" for short) distribution, for the first time the KUMOFW distribution is introduced and studied. We present some structural properties of the proposed distribution, including explicit expressions for the moments. The method of maximum likelihood is used to estimate the model parameters. We illustrate the importance of the new model by means of application to real data set.

Keywords: Kumaraswamy-Marshal-Olkin generalization distribution, Flexible Weibull Distribution, Estimation Process, Kurtosis, Skewness, MLE, Reliability.

1 Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with revelent statistical methodologies. In fact, the statistics literature is filled with hundreds of continuous univariate distributions. However, in recent years, applications from the environmental, financial, biomedical sciences, engineering among others, have further shown that data sets following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions a significant progress has been made toward the generalization of some well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

2 The Flexible Weibull Distribution

A random variable T is said to follow the Flexible Weibull distribution with two parameters if the probability density function pdf of T is as follows:

$$f(x,\alpha,\beta)=\left(\alpha+\frac{\beta}{x^2}\right)\,exp\left(\alpha\,x-\frac{\beta}{x}\right)\,exp\left(-e^{\alpha\,x-\frac{\beta}{x}}\right)\quad x,\alpha,\beta>0 \tag{1}$$

The parameters α and β are usually called the shape and scale parameters, respectively. The statistical properties of the density are obtained; However, The estimation of the unknown parameters is easier to calculate.

The cumulative distribution function cdf of the two parameters Flexible Weibull distribution of the random variable T is given by:

$$F(x,\alpha,\beta) = 1 - exp\left(-e^{\alpha x - \frac{\beta}{x}}\right) \qquad x,\alpha,\beta > 0 \tag{2}$$

The corresponding Survival function $S(t)$ and the hazard rate function $h(x)$ are:

$$S(x,\alpha,\beta) = exp\left(-e^{\alpha x - \frac{\beta}{x}}\right) \qquad x,\alpha,\beta > 0 \tag{3}$$

$$h(x,\alpha,\beta) = \left(\alpha + \frac{\beta}{x^2}\right) exp\left(-e^{\alpha x - \frac{\beta}{x}}\right) \quad x,\alpha,\beta > 0 \tag{4}$$

3 The Kumaraswamy Marshal-Olkin Family of Distributions

A random variable X has the cdf of the Kumaraswamy Marshal-Olkin (''KUMO'') family of distributions by

$$F_{kwMO}(x;a,b,p) = 1 - \left\{1 - \left(\frac{G(x;\xi)}{1-p\bar{G}(x;\xi)}\right)^a\right\}^b \tag{5}$$

where $a > 0$; $b > 0$, $p = 1 - \bar{p}$ and $\bar{p} > 0$ are three additional shape parameters. $\bar{G}(x;\xi) = 1 - G(x;\xi)$. For each baseline G, the ''KUMO-G'' cdf is given by (5). The density function corresponding to (8) is given by

$$f_{kwMO}(x;a,b,p) = \frac{ab(1-p)g(x;\xi)G(x;\xi)^{a-1}}{(1-p\bar{G}(x;\xi))^{a+1}}\left\{1 - \left(\frac{G(x;\xi)}{1-p\bar{G}(x;\xi)}\right)^a\right\}^{b-1} \tag{6}$$

Eq. (6) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions. Hereafter, a random variable X with density function (6) is denoted by $X \sim KwMO(a,b,p,\xi)$. The hrf of X becomes

$$h_{kwMO}(x;a,b,p) = \frac{ab(1-p)g(x;\xi)G(x;\xi)^{a-1}}{[1-p\bar{G}(x;\xi)]\{[1-p\bar{G}(x;\xi)]^{a}-G(x;\xi)^{a}\}} \qquad (7)$$

In this context, we propose an extension of the Lindley-Lomax distribution based on the family of Kumaraswamy-Marshal Olkin (denoted with the prefix KUMOLL for short) distribution. Kumaraswamy-Marshal Olkin distribution, introduced by (Gauss M. Cordeiro et al., 2015). They are proposed a new extension of the MO family for a given baseline distribution with cdf$G(x;\xi)$, survival function $\bar{G}(x;\xi) = 1 - G(x;\xi)$and pdf $g(x;\xi)$ depending on a parameter ξ.

The rest of this article is organized as follows: in Section 4, we introduce the new defined distribution and investigate its basic properties, including the shape properties of its density function and the hazard rate function, moments and measurements based on the moments. Section 7 discusses the estimation of parameters by the method of maximum likelihood. An application of real survival data illustrated in Section 8. Our work concluded in Section 9.

4. The Kumaraswamy Marshall-Olkin Flexible Weibull Distribution

A random variable X has the cdf of the Kumaraswamy Marshal-Olkin Flexible Weibull (''KUMOFW'') family of distributions by

$$F_{KUMOFW}(x;\underline{\tau}) = 1 - \left\{1 - \left(\frac{1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}{1-p\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}\right)^{a}\right\}^{b} \qquad (8)$$

Where $\underline{\tau} = (a,b,p,\alpha,\beta)$ and $(a,b,p\ and\ \alpha) > 0$ are non-negative shape Parameters, $\beta > 0$ is positive scale parameter, $p = 1-\bar{p}$ and $\bar{p} > 0$ is the tilt parameter. The corresponding pdf and Survival function,

$$f_{KUMOFW}(x;\underline{\tau}) = ab(1-p)\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\left\{1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\right\}^{a-1}$$

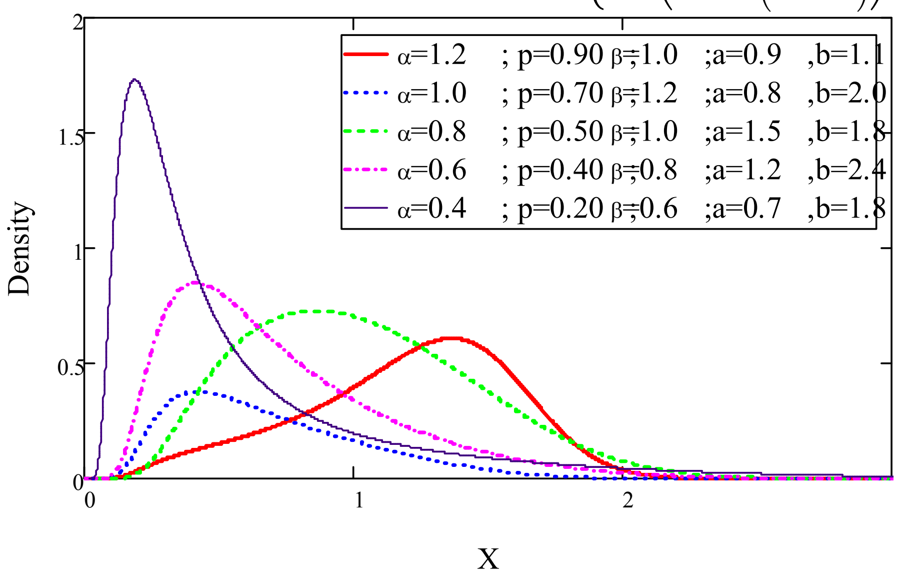
$$\left(1-p\left[1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\right]\right)^{-(a+1)}\left\{1-\left(\frac{1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}{1-p\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}\right)^{a}\right\}^{b-1} \qquad (9)$$

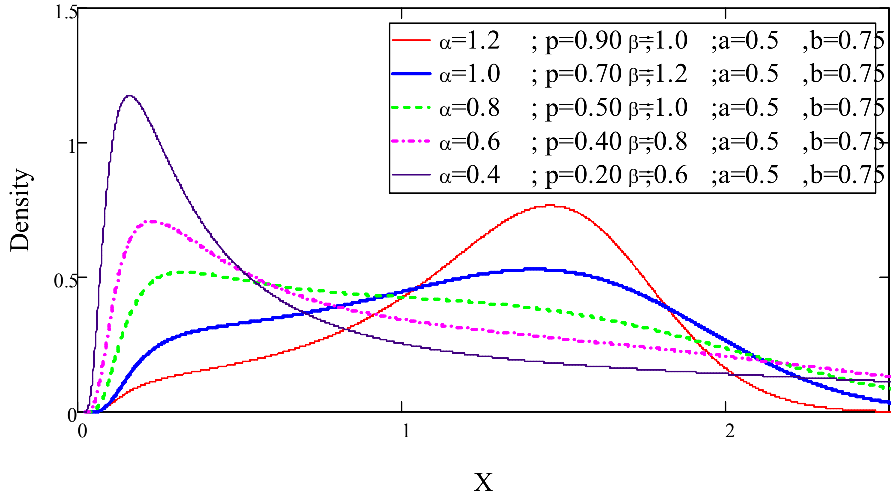
$$S_{KUMOFW}(x;\underline{\tau}) = 1-F_{KUMOFW}(x;\underline{\tau})$$

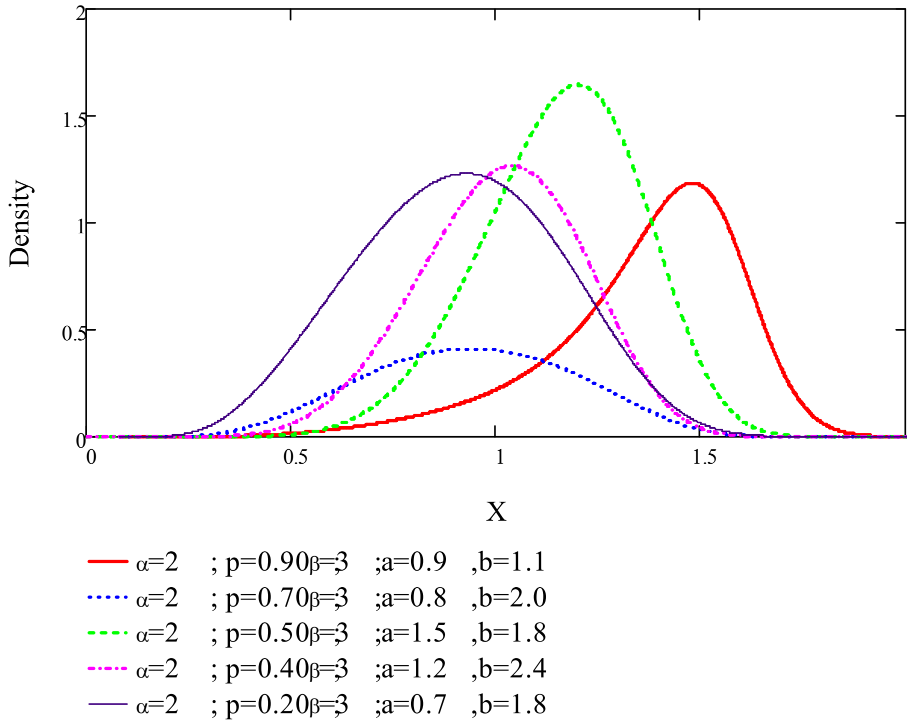
$$=\left\{1-\left(\frac{1-\exp(-e^{\alpha x-\frac{\beta}{x}})}{1-p\exp(-e^{\alpha x-\frac{\beta}{x}})}\right)^{a}\right\}^{b}$$

$$h_{KUMOFW}(x;\underline{\tau}) = ab(1-p)\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\left\{1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\right\}^{a-1}$$

$$\left(1-p\left[1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\right]\right)^{-(a+1)}\left\{1-\left(\frac{1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}{1-p\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}\right)^{a}\right\}^{-1} \qquad (10)$$




Density
X
α=1.2 ; p=0.90 β=1.0 ;a=0.5 ,b=0.75
α=1.0 ; p=0.70 β=1.2 ;a=0.5 ,b=0.75
α=0.8 ; p=0.50 β=1.0 ;a=0.5 ,b=0.75
α=0.6 ; p=0.40 β=0.8 ;a=0.5 ,b=0.75
α=0.4 ; p=0.20 β=0.6 ;a=0.5 ,b=0.75


Density
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α=2 ; p=0.40 β=3 ;a=1.2 ,b=2.4
α=2 ; p=0.20 β=3 ;a=0.7 ,b=1.8

Figure 1: Plots of the $KUMOFW$ density for selected parameter values.

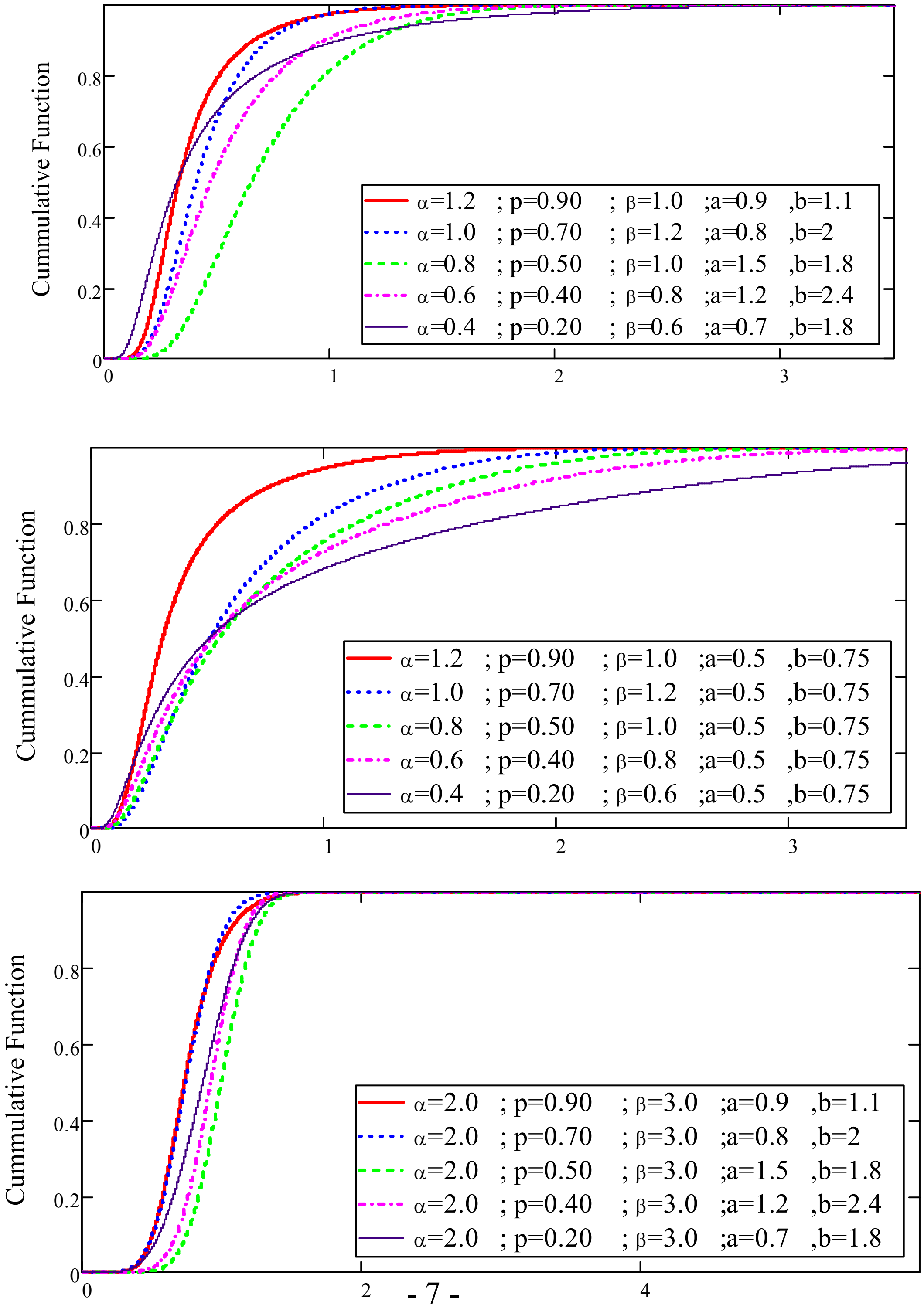
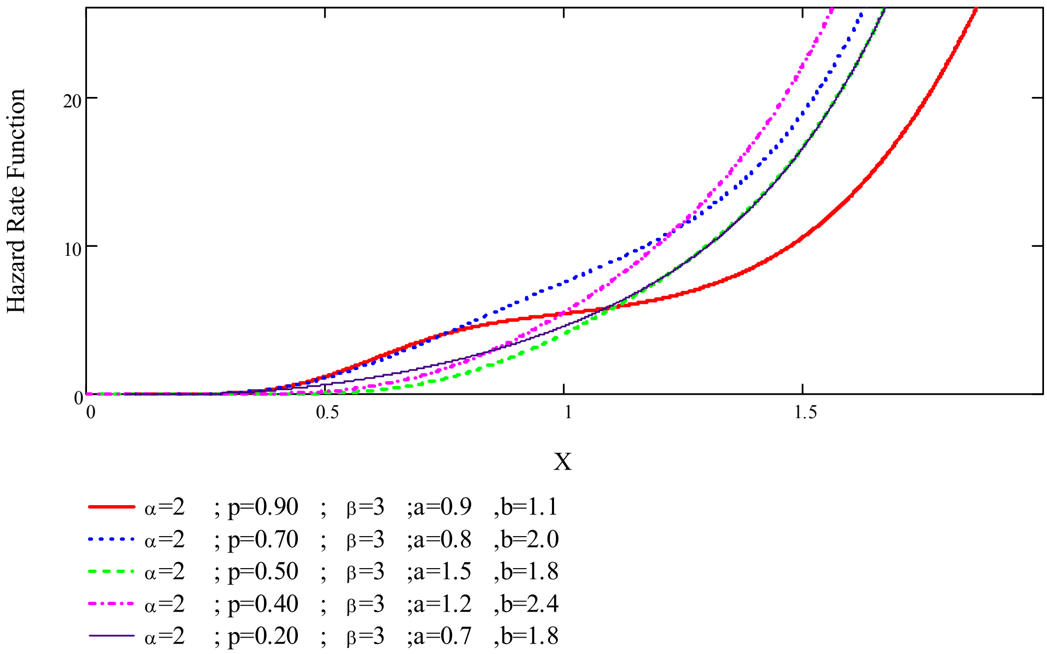
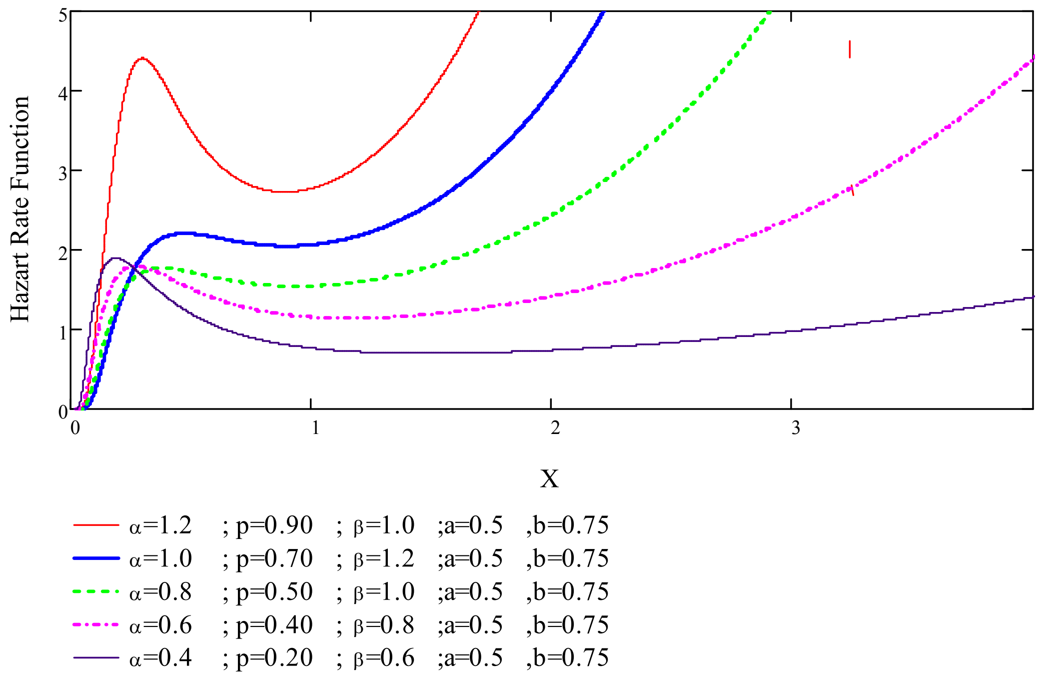


Figure 2: Plots of the $KUMOFW$ cdf for selected parameter values.





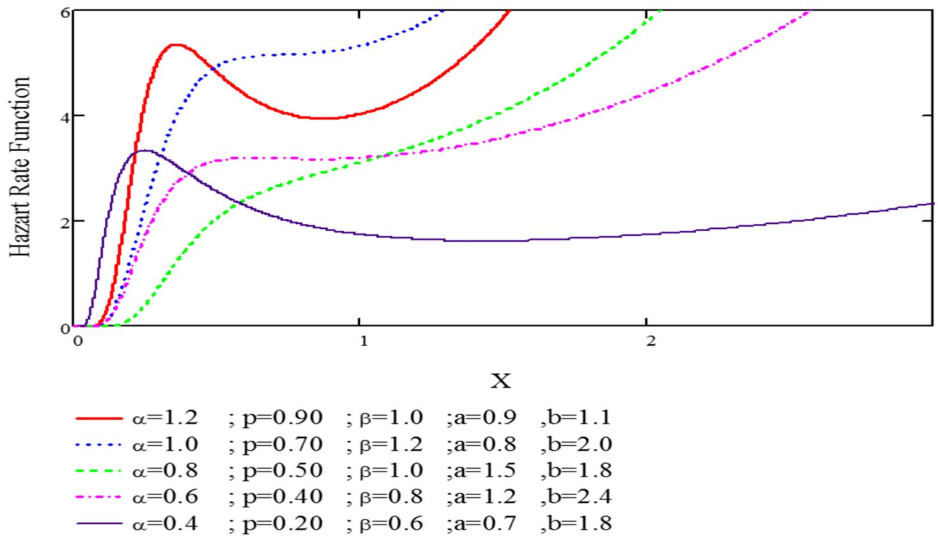


Figure 3: Plots of the $KUMOFW$ hazard rate function for selected parameter values.

5. Special Distributions

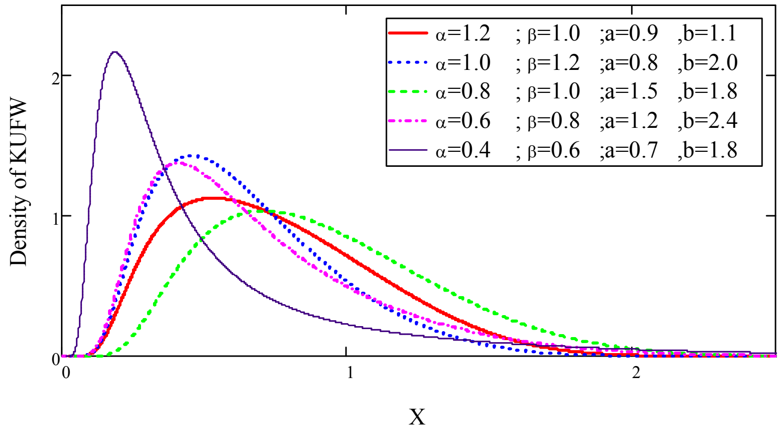
The following well-known and new distributions are special sub-models of the $KUMOFW$ distribution.

- **Kumaraswamy Flexible Weibull Distribution**

If $p = 0$, the $KUMOFW$ distribution reduces to

$$f_{KUFW}(x;\underline{\tau}) = ab(1-p)\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}} \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\left\{1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right\}^{a-1} \left\{1 - \left(1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right)^{a}\right\}^{b-1} \quad (11)$$

which is the Kumaraswamy Flexible Weibull Distribution (KUFW).for $a = b = 1$ we obtain the Flexible Weibull distribution.



- **Marshal-Olikin Flexible Weibull Distribution**

If $a = b = 1$, the KUMOFW distribution reduces to

$$f_{MOFW}(x;\underline{\tau}) = \frac{(1-p)\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}{\left(1-p\left[1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\right]\right)^2} \tag{12}$$

which is Marshal-Olkin Flexible Weibull Distribution (MOFW).for, $p = 0$ we obtain the Flexible Weibull distribution.

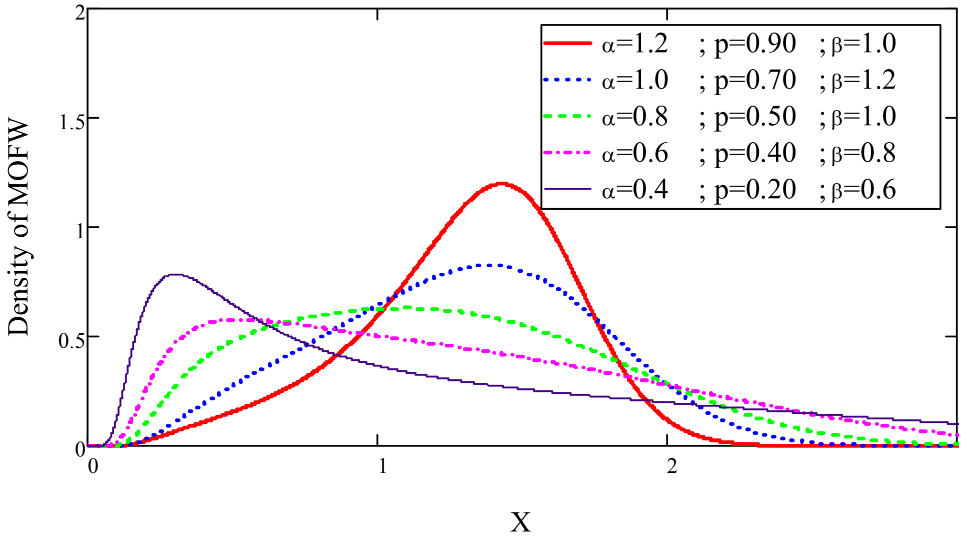


Table1. Some Special Distributions

Distribution	**a**	***b***	***p***	***α***	**β**
KU Flexible Weibull	--	--	0	--	--
MO Flexible Weibull	1	1	--	--	--
Exponentiated MO Flexible Weibull	--	1	--	--	--
Exponentiated Flexible Weibull	--	1	0	--	--
Flexible Weibull	1	1	0	--	--
The proportional reversed hazard rate model for Flexible Weibull	--	1	0	--	--
The proportional hazard rate model Exponentiated Flexible Weibull	1	--	0	--	--

6. Expansions for the density function

In this section, we will derive a useful expansion for the KUMOFW "pdf" for brevity of notation,by using properties of exponentiated distribution expanding the binomial theorem

$$[1-z]^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)\ \ j!} z^{j} \tag{13}$$

In equation (9), we can write

$$f_{KUMOFW}(x;\underline{\tau}) = ab(1-p)\left(\alpha+\frac{\beta}{x^2}\right)\sum_{j=0}^{\infty}\sum_{t=0}^{\infty}\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{u=0}^{\infty} p^{t+u} e^{\alpha x-\frac{\beta}{x}} \left[\exp(-e^{\alpha x-\frac{\beta}{x}})\right]^{j+l+u}$$

$$(-1)^{j+t+y+l+u}\binom{a-1}{j}\binom{-(a+1)}{t}\binom{b-1}{y}\binom{t+ay+1}{l}\binom{-ay+1}{u} \tag{14}$$

7. Properties of Marshall-Olkin Flexible Weibull Distribution

In this section will present some of our density properties by following (9) we obtain:

6.1 Moments

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

Here and henceforth, let X be a $KUMOFW$ random variable following (9). The r^{th} moment of X can be obtained from (9) as

$$\mu_r' = \int_0^\infty x^r \; f_{kwMOFW}(x;\underline{\tau})\, dx$$

$$\mu_r' = \eta_{j,t,y,i,u} \int_0^\infty x^r \; \left(\alpha+\frac{\beta}{x^2}\right) e^{\alpha x-\frac{\beta}{x}} \left[\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)\right]^{j+i+u} dx \qquad (15)$$

where

$$\eta_{j,t,y,i,u}(a,b,p) = ab(1-p)\sum_{j,t,y,i,u=0}^{\infty} (-1)^{j+t+y+i+u}\binom{a-1}{j}\binom{-(a+1)}{t}\binom{b-1}{y}\binom{t+ay+1}{i}\binom{1-ay}{u}$$

For $(j+i+u)>0$ real integer, we obtain

$$\left[\exp(-e^{\alpha x-\frac{\beta}{x}})\right]^{j+i+u} = \sum_{v=0}^{\infty} \frac{(-1)^v\,(j+l+u)^v}{v!}\; e^{v(\alpha x-\frac{\beta}{x})} \qquad (16)$$

Substituting (16) in equation (15) gives

$$µ_r' = \omega_v \int_0^\infty x^r \ \ (\alpha + \beta x^{-2}\ ) e^{(1+v)\alpha x}\ e^{-(1+v)\frac{\beta}{x}}\ dx$$

(17)

where

$$\omega_v = \eta_{j,t,y,i,u}(a,b,p) \sum_{v=0}^{\infty} \frac{(-1)^v\ (j+l+u)^v}{v!}$$

We now use the Taylor series expansion for the exponential function given by

$$e^{-(1+v)\frac{\beta}{x}} = \sum_{m=0}^{\infty} \frac{(-1)^m\ (1+v)^m\ (\beta)^m\ \ (x)^{-m}}{m!}$$

(18)

Substituting (18) in equation (17) gives

$$µ_r' = \omega_v p^{t+u} \sum_{m=0}^{\infty} \frac{(-1)^m\ (1+v)^m\ (\beta)^m\ \ (x)^{-m}}{m!} \int_0^\infty x^{r-m}\ \ (\alpha + \beta x^{-2}\ ) e^{(1+v)\alpha x}\ \ dx$$

$$µ_r' = v(x_i, a, b, p, \beta) \left[\frac{\alpha\ \Gamma(r-m+1)}{\alpha^{r-m+1}(\mathrm{v}+1)^{r-m+1}} + \frac{\beta\ \Gamma(r-m-1)}{\alpha^{r-m-1}(\mathrm{v}+1)^{r-m-1}} \right]$$

(19)

where

$$v(x_i, a, b, p, \beta) = \omega_v p^{t+u} \sum_{m=0}^{\infty} \frac{(-1)^m \, (1+v)^m \, (\beta)^m \, (x)^{-m}}{m!}$$

6.2 Moment Generating Function

Let X be a random variable having the KUMOFW density function (9). We now derive a closed form expression for the mgf, say $\mathrm{M_x(t)} = E[exp(tx)]$, of X.

$$M_x(t) = \int_0^{\infty} exp(tx) \;\; f_{KUMOFW}(x; \underline{\tau}) \; dx$$

(20)

Expanding the exponential in Taylor series, we have

$$exp(tx) = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r \;\; f_{KUMOFW}(x; \underline{\tau}) \; dx$$

$$M_x(t) = \Upsilon(x_i, a, b, p, \beta) \left[\frac{\alpha \; \Gamma(r-m+1)}{\alpha^{r-m+1} (\mathrm{v}+1)^{r-m+1}} + \frac{\beta \; \Gamma(r-m-1)}{\alpha^{r-m-1} (\mathrm{v}+1)^{r-m-1}} \right]$$

(21)

where

$$\Upsilon(x_i, a, b, p, \beta) = \eta_{j,t,y,i,u}(a, b, p) p^{t+u} \sum_{m,r,v=0}^{\infty} \frac{(-1)^m \, (1+v)^m \, (\beta)^m \;\, t^r \, (x)^{-m}}{v! \, m! \;\, r! \, (-1)^{-v} (j+l+u)^{-v}}$$

$$\eta_{j,t,y,i,u}(a, b, p) = \; ab(1-p) \sum_{j,t,y,i,u=0}^{\infty} (-1)^{j+t+y+i+u} \binom{a-1}{j} \binom{-(a+1)}{t} \binom{b-1}{y} \binom{t+ay+1}{i} \binom{1-ay}{u}$$

The moment generating function of the $KUMOFW$ distribution in (20) and in equations (21) are the main results of r^{th} moment multipling with Taylor factor $\sum_{r=0}^{\infty}\frac{t^r}{r!}$, $M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\ \mu_r'$

6.3 Quantile function and simulation

We present a method for simulating from the $KUMOFW$ distribution (9). Using the method of inversion we can generate random numbers from $KUMOFW$, then, the quantile function corresponding to (8) where u ~ U(0, 1) is

$$u = 1 - \left\{1 - \left(\frac{1-\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}{1-p\exp\left(-e^{\alpha x-\frac{\beta}{x}}\right)}\right)^{a}\right\}^{b}$$

After simple calculation this yields

$$X_u = \frac{\mathrm{N}(u) \pm \sqrt{\mathrm{N}(u)^2 + 4\alpha\beta}}{2\alpha} \tag{22}$$

where

$$\mathrm{N}(u) = \log\left[\log\left(\frac{1-p\left[1-\{1-(u)^{1/b}\}^{1/a}\right]}{1-[1-\{1-(u)^{1/b}\}^{1/a}]}\right)\right]$$

One can use equation (22) to generate random numbers when the parameters a, b, p, α and β are known.

6.4 Skewness and Kurtosis

The shortcomings of the classical kurtosis measure are well-known. There are many heavy-tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical kurtosis for many of the Kumaraswamy distributions.

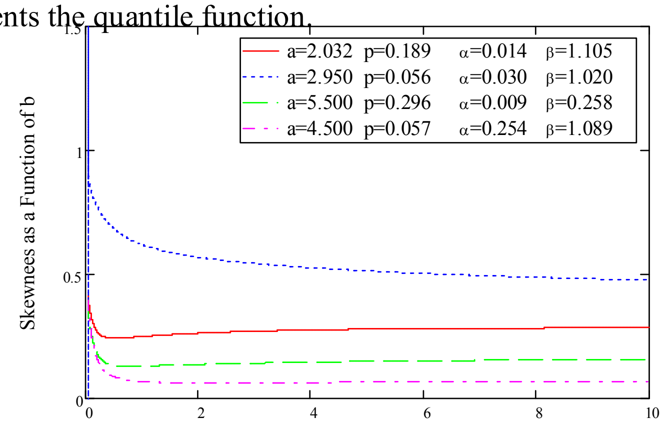
The Bowley's skewness (see Kenney and Keeping, 1962) is based on quartiles:

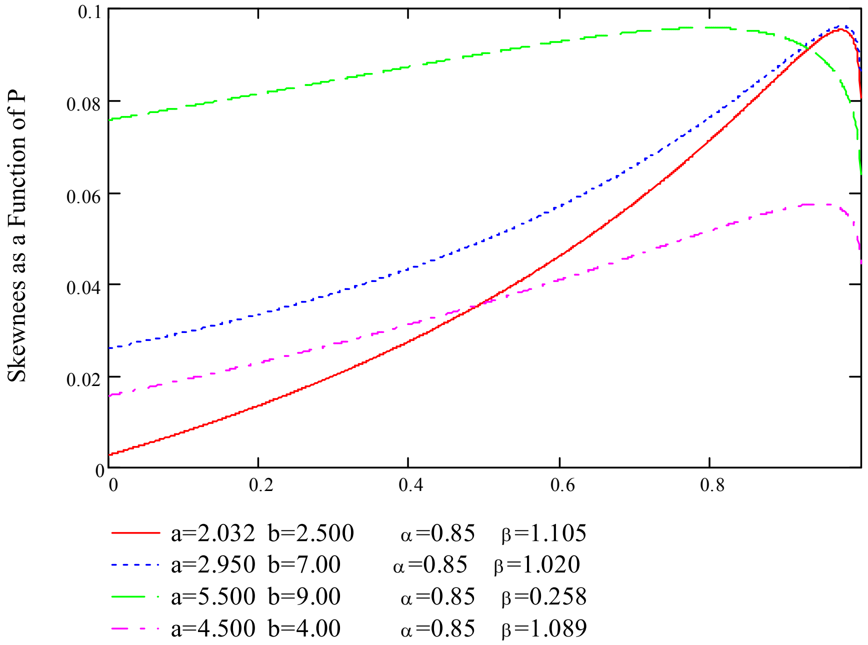
$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

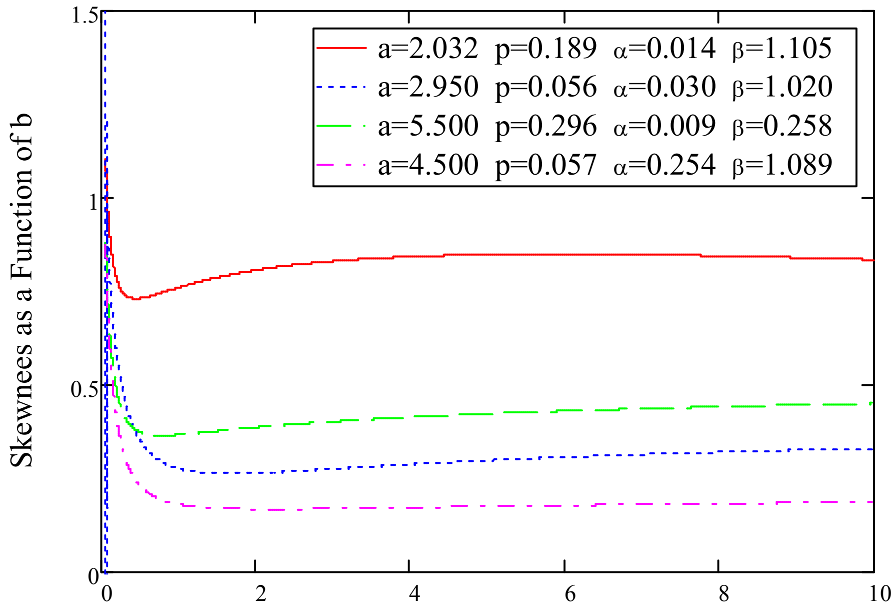
and the Moors' kurtosis (see Moors, 1998) is based on octiles:

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}$$

where Q(·) represents the quantile function.




Skewnees as a Function of P
0.1
0.08
0.06
0.04
0.02
0
0
0.2
0.4
0.6
0.8
a=2.032 b=2.500 α=0.85 β=1.105
a=2.950 b=7.00 α=0.85 β=1.020
a=5.500 b=9.00 α=0.85 β=0.258
a=4.500 b=4.00 α=0.85 β=1.089


Skewnees as a Function of b
1.5
1
0.5
0
0
2
4
6
8
10
a=2.032 p=0.189 α=0.014 β=1.105
a=2.950 p=0.056 α=0.030 β=1.020
a=5.500 p=0.296 α=0.009 β=0.258
a=4.500 p=0.057 α=0.254 β=1.089

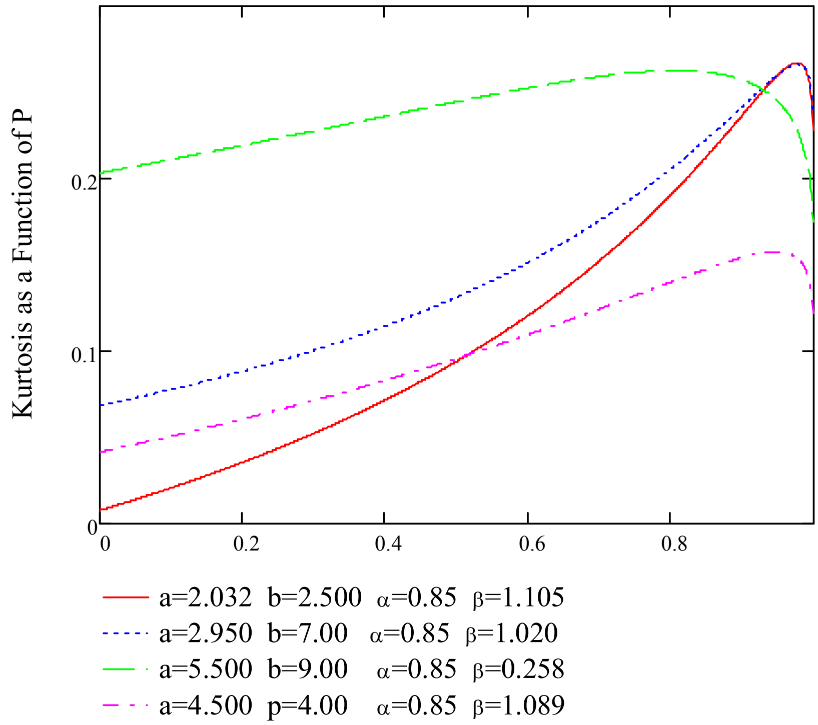


Figure 4. Plots of skewness and kurtosis for different value of parameters.

Plots of the skewness and kurtosis for some choices of the parameters as function of b, and for some choices of the parameters as function of P, for different value of β andα, are shown in Figure 4.we can observe that the plots of the skewness and kurtosis decrease when b increases for fixed a and when a increases for fixed b.

6.5 Order Statistics

The density function $f_{r:n}(x)$ of the i^{th} order statistic,$for\ i = 1,\ldots,n$, from random variables $X_1,\ldots, X_n$ having density (9), is given by

$$f_{r:n}(x;\underline{\tau}) = \frac{1}{\beta_{(r,n-r+1)}} f(x;\underline{\tau})[F(x;\underline{\tau})]^{r-1}[1-F(x;\underline{\tau})]^{n-r} \quad (22)$$

For $n > 0$ non-real integer, we obtain

$$\left[1-F\left(x;\underline{\tau}\right)\right]^{n-r}=\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i}\left[F\left(x;\underline{\tau}\right)\right]^{i} \tag{23}$$

Substituting (23) in equation (22) gives

$$f_{r:n}\left(x;\underline{\tau}\right)=\frac{1}{\beta(r,n-r+1)}\,f\left(x;\underline{\tau}\right)\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i}\left[F\left(x;\underline{\tau}\right)\right]^{i+r-1}$$

$$f_{r:n}\left(x;\underline{\tau}\right)=\sum_{i=0}^{n-r}\frac{(-1)^{i}\quad n!}{i!\;\;(r-1)!\;\;(n-r-1)!}\left[F\left(x;\underline{\tau}\right)\right]^{i+r-1}f\left(x;\underline{\tau}\right) \tag{24}$$

7 Estimation and Information Matrix

In this section, we discuss maximum likelihood estimation and inference for the KUMOFW distribution. Let $x_1, x_2, \ldots, x_n$ be a random sample from $X \sim KwMoFW(\underline{\tau})$. The log-likelihood function for $\underline{\tau}=(a,b,p,\alpha,\beta)$ written as:

$$\begin{aligned}\ell(\underline{\tau}) &= n\log a+n\log b+n\log(1-p)+\sum_{i=1}^{n}\left[\log\left(\alpha+\frac{\beta}{x^{2}}\right)\right]\\ &+\sum_{i=1}^{n}[\log v(x_i)]+\sum_{i=1}^{n}\log\omega(x_i)+(a-1)\sum_{i=1}^{n}\log[1-\omega(x_i)]\\ &-(a+1)\sum_{i=1}^{n}\log\{1-p[1-\omega(x_i)]\}+(b-1)\sum_{i=1}^{n}\log\left\{1-\left[\frac{1-\omega(x_i)}{1-p\omega(x_i)}\right]^{a}\right\}\end{aligned} \tag{25}$$

The score vector $U\left(\underline{\tau}\right)=\left(\frac{\partial\ell}{\partial a}\;\frac{\partial\ell}{\partial b}\;\frac{\partial\ell}{\partial p}\;\frac{\partial\ell}{\partial\alpha}\;\frac{\partial\ell}{\partial\beta}\right)^{T}$, where the components corresponding to the parameters in $\underline{\tau}$ are given by differentiating (25) . by setting $\omega(x_i)=\exp(-e^{\alpha x-\frac{\beta}{x}})$, $v(x_i)=e^{\alpha x-\frac{\beta}{x}}$ and $n(x_i)=\frac{1-\omega(x_i)}{1-p\omega(x_i)}$

$$\frac{\partial \ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log[1-\omega(x_i)]-\sum_{i=1}^{n}\log\{1-p[1-\omega(x_i)]\}-(b-1)\sum_{i=1}^{n}\frac{n(x_i)^a\log n(x_i)}{1-n(x_i)^a}$$

$$\frac{\partial \ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log(1-n(x_i)^a)$$

$$\frac{\partial \ell}{\partial p}=\frac{-n}{1-p}+(a+1)\sum_{i=1}^{n}\frac{1-\omega(x_i)}{1-p[1-\omega(x_i)]}+(b-1)\sum_{i=1}^{n}\frac{-a\,n(x_i)^{a-1}\omega(x_i)[1-\omega(x_i)]}{[1-n(x_i)^a][1-p\omega(x_i)]^2}$$

$$\frac{\partial \ell}{\partial \alpha}=\sum_{i=1}^{n}\frac{1}{\alpha+\frac{\beta}{{x_i}^2}}+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}x_i\,v(x_i)+(a-1)\sum_{i=1}^{n}\frac{(x_i)\,\omega(x_i)v(x_i)}{(1-\omega(x_i))}$$

$$+p(a+1)\sum_{i=1}^{n}\frac{(x_i)\,\omega(x_i)v(x_i)}{1-p[1-\omega(x_i)]}-a(b-1)\sum_{i=1}^{n}\frac{n(x_i)^{a-1}\frac{\partial n(x_i)}{\partial \alpha}}{1-n(x_i)^a}$$

where (26)

$$\frac{\partial n(x_i)}{\partial \alpha}=\frac{[1-p\omega(x_i)][(x_i)\,\omega(x_i)\,v(x_i)]-[1-\omega(x_i)][p(x_i)\,\omega(x_i)\,v(x_i)]}{[1-p\omega(x_i)]^2}$$

$$\frac{\partial \ell}{\partial \beta}=\sum_{i=1}^{n}\frac{1/{x_i}^2}{\alpha+\frac{\beta}{{x_i}^2}}+\sum_{i=1}^{n}\frac{1}{x_i}-\sum_{i=1}^{n}\frac{1}{x_i}v(x_i)-(a-1)\sum_{i=1}^{n}\frac{1}{(x_i)}\frac{\omega(x_i)v(x_i)}{[1-\omega(x_i)]}$$

$$-p(a+1)\sum_{i=1}^{n}\frac{1}{(x_i)}\frac{\omega(x_i)v(x_i)}{1-p[1-\omega(x_i)]}-a(b-1)\sum_{i=1}^{n}\frac{n(x_i)^{a-1}\frac{\partial n(x_i)}{\partial \beta}}{1-n(x_i)^a}$$

Where

$$\frac{\partial n(x_i)}{\partial \beta} = \frac{[1-p\omega(x_i)]\left[\frac{1}{(x_i)}\,\omega(x_i)\,v(x_i)\right]+[1-\omega(x_i)]\left[p\,\frac{1}{(x_i)}\,\omega(x_i)\,v(x_i)\right]}{[1-p\omega(x_i)]^2}$$

The maximum likelihood estimates (MLEs) of the parameters are the solutions of the nonlinear equations (26), $\nabla\ell = 0$ which are solved iteratively, these solutions will yield the ML estimators for$(\hat{a},\hat{b},\hat{p},\hat{\alpha}\ and\ \hat{\beta})$. For the parameters KWMOFW distribution, all the second order derivatives exist. Thus we require the 5×5 unit observed information matrix

$$J = J(\underline{\tau}) = \begin{bmatrix} j_{aa} & j_{ab} & j_{ap} & j_{a\alpha} & j_{a\beta} \\ j_{ba} & j_{bb} & j_{bp} & j_{b\alpha} & j_{b\beta} \\ j_{pa} & j_{pb} & j_{pp} & j_{p\alpha} & j_{p\beta} \\ j_{\alpha a} & j_{\alpha b} & j_{\alpha p} & j_{\alpha\alpha} & j_{\alpha\beta} \\ j_{\beta a} & j_{\beta b} & j_{\beta p} & j_{\beta\alpha} & j_{\beta\beta} \end{bmatrix} \quad (27)$$

Using MATHCAD software to solve the inverse dispersion matrix analytically, these solutions will yield asymptotic variance and covariance of these ML estimators for $(\hat{a},\hat{b},\hat{p},\hat{\alpha}\ and\ \hat{\beta})$ and showing in table 5, using (27), we approximate $100(1-\gamma)$ percentage, confidence intervals for are determined respectively as

$$\hat{a} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{j_{aa}}} \qquad \hat{b} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{j_{bb}}} \qquad \hat{p} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{j_{pp}}}$$

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{j_{\alpha\alpha}}} \qquad \hat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{j_{\beta}}}$$

8 Empirical Application

In this section we use a real data set of a KUMOFW distribution. parameters are estimated via the MLE method described in Section (7). Using the MATHCAD software. First we describe the data sets. Then we report the the MLEs (and the corresponding standard errors in parentheses) of the parameters and we shall apply formal goodness-of-fit tests to verify which distribution fits better the real data sets with other various distributions including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC). In general, the smaller values of these statistics is the better fit to the data. We shall compare the proposed KUMOFW distribution with several other lifetime distributions as beta-Weibull (BW) [5], the Kumaraswamy-Weibull (KUW) [7], the Marshall-Olkin extended Weibull (MOW) [6] distributions . Finally, we perform the Kolmogorov-Smirnov (K-S) statistic and $-2\ell(\hat{\theta})$ tests .

"Data of carbon fibers"

The real data set was originally reported by Badar and Priest [10], which represents the strength measured in GPa for single carbon fibers and impregnated at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 100 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. Here, we consider the data set of single fibers of 20 mm in gauge with a sample of size 63. The data are:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445,
2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618,

2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Table 2: MLEs of the model parameters (standard errors in parentheses)and the statistics AIC, BIC and CAIC for Carbon Fibers

Distributions	Parameters					Statistic		
	$\hat{a}$	$\hat{b}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	AIC	BIC	CAIC
KwMOFW	**6.155**	**111.536**	**0.007**	**0.121**	**2.205**	**111.714**	**110.71**	**112.766**
	(0.573)	**(16.98)**	**(0.074)**	**(0.019)**	**(0.295)**			
MOFW	**1**	**1**	**0.293**	**0.115**	**2.148**	**153.598**	**152.595**	**154.651**
	----	**----**	**(3.71)**	**(20.698)**	**(2.152)**			
Flexible Weibull	**1**	**1**	**1**	**0.115**	**2.155**	**289.691**	**288.687**	**290.743**
	----	**----**	**----**	**(0.012)**	**(0.105)**			
KwMO-Fréchet	**0.053**	**1.046**	**0.0203**	**----**	**4.066**	**121.867**	**132.583**	**122.920**
	(0.084)	**(0.901)**	**(0.054)**	**----**	**(1.941)**			
Beta-Fréchet (BFr)	**12.847**	**20.762**	**----**	**----**	**1.167**	**120.594**	**129.166**	**121.283**
	(91.822)	**(64.327)**	**----**	**----**	**(1.926)**			
Exponentiated-Fréchet (EFr)	**----**	**----**	**----**	**7.031**	**2.364**	**118.700**	**125.130**	**119.107**
	----	**----**	**----**	**(8.504)**	**(1.027)**			
Marshall-Olkin Extended Fréchet (MOFr)	**----**	**----**	**10.343**	**----**	**7.906**	**119.746**	**126.175**	**120.153**
	----	**----**	**(12.421)**	**----**	**(1.142)**			
Fréchet (Fr)	**----**	**----**	**----**	**----**	**5.433**	**121.804**	**126.091**	**122.004**
	----	**----**	**----**	**----**	**(0.508)**			

Table 3: Covariance of ML estimators for $(\hat{a}, \hat{b}, \hat{p}, \hat{\alpha}\ and\ \hat{\beta})$

$Cov(\hat{a},\hat{b}) = cov(\hat{b},\hat{a}) = -0.261$	$Cov(\hat{p},\hat{\beta}) = cov(\hat{\beta},\hat{p}) = 5.569 \times 10^{-3}$
$Cov(\hat{a},\hat{p}) = cov(\hat{p},\hat{a}) = -0.028$	$Cov(\hat{\alpha},\hat{\beta}) = cov(\hat{\beta},\hat{\alpha}) = -4.057 \times 10^{-3}$
$Cov(\hat{a},\hat{\alpha}) = cov(\hat{\alpha},\hat{a}) = 2.499 \times 10^{-4}$	
$Cov(\hat{a},\hat{\beta}) = cov(\hat{\beta},\hat{a}) = 0.102$	
$Cov(\hat{b},\hat{p}) = cov(\hat{p},\hat{b}) = -0.412$	
$Cov(\hat{b},\hat{\alpha}) = cov(\hat{\alpha},\hat{b}) = -0.113$	
$Cov(\hat{b},\hat{\beta}) = cov(\hat{\beta},\hat{b}) = 0.368$	
$Cov(\hat{p},\hat{\alpha}) = cov(\hat{\alpha},\hat{p}) = 4.637 \times 10^{-4}$	

Table 2, lists the MLEs (and the corresponding standard errors in parentheses) of the parameters of all the models and the statistics AIC, BIC and CAIC for survival times (in years) of a group of patients data set, we notice that the proposed **KUMOFW** model presents the smallest values of the statistics AIC, BIC and CAIC, and hence should be chosen as the best model among all the distributions to fit the data set.

Table 4: the Kolmogorov-Smirnov (K-S) statistic and $-2\ell(\hat{\theta})$.

Data	Model	**KUMOFW**	**MOFW**	**Flexible Weibull**
Carbon Fibers	$K-S$	0.984	0.766	1.00
	$-2\ell(\hat{\theta})$	101.714	143.598	279.691

9 Concluding Remarks

The well-known generalized Pareto distribution, is extended by introducing three extra shape parameters, thus defining the Kumaraswamy Marshal Olikin Flexible Weibull distribution (KUMOFW) having a broader class of hazard rate and density functions. This is achieved by taking (5) as the baseline cumulative distribution of the Kumaraswamy Marshal Olikin distribution. A detailed study on the mathematical properties of the new distribution is presented. The new model includes as special sub-models as Marshal Olikin Flexible Weibull distribution (MOFW) , Flexible Weibull (FW) and Weibull distributions. The estimation of the model parameters is approached by maximum likelihood and the observed information matrix is obtained. An application to a real data set indicates that the fit of the new model is superior to the fits of its principal sub-models. We hope that the proposed model may be interesting for a wider range of statistical research.

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Bayesian Estimation and Prediction for the Generalized Burr Distribution

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Abstract

In this paper, Bayes estimators for the parameters, reliability and hazard rate functions of the generalized Burr distribution are derived. Point and credible interval estimation are considered based on Type II censored data under a symmetric and asymmetric loss functions. Also, Bayesian prediction for a future observation is obtained using two-sample prediction technique. Finally, numerical examples are given via Markov Chain Monte Carlo simulation study and some interesting comparisons are presented to illustrate the theoretical results. Moreover, the results are applied on real data sets.

Keywords: *Generalized Burr distribution; loss functions; Type II censored data; Bayesian prediction; Bayesian predictive density function; Markov Chain Monte Carlo simulation.*

1. Introduction

Kibria and Shakil (2011) introduced the five-parameter family of Burr Type distributions based on the generalized Pearson differential equation. This family is considered more flexible and a natural generalization of the Burr, *generalized beta Type II* (GBII) and also other distributions. They referred to this distribution as the *generalized Burr* (GBurr) distribution. It is observed that the proposed distribution is skewed to the right and have most of the properties of skewed distributions.

The *probability density function* (pdf) and *cumulative distribution function* (cdf) of the GBurr($\underline{\theta}$) distribution are given, respectively, by

$$f(x;\underline{\theta}) = \varphi(\underline{\theta})\ x^{\theta_4-1}\left[1+\frac{\theta_5}{\theta_2}\,x^{\theta_1}\right]^{-\theta_3}, \qquad x>0;\ \ (\underline{\theta}>\underline{0}), \tag{1}$$

and

$$F(x;\underline{\theta}) = \varphi(\underline{\theta})\sum_{l=0}^{\infty}\left[\frac{(-1)^l\binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l\ x^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right], \qquad x>0;\ \ (\underline{\theta}>\underline{0}), \tag{2}$$

where

$$\varphi(\underline{\theta}) = \frac{\theta_1\left(\frac{\theta_5}{\theta_2}\right)^{\left(\frac{\theta_4}{\theta_1}\right)}}{B\left(\frac{\theta_4}{\theta_1},\theta_3-\frac{\theta_4}{\theta_1}\right)}, \qquad \underline{\theta}=(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)' \tag{3}$$

and $B(.,.)$ denotes the beta function.

The *reliability function* (rf) and *hazard rate function* (hrf) are given, respectively, by

$$R(x) = 1 - \left(\varphi(\underline{\theta})\sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l x^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right),$$

$$x > 0;\ (\underline{\theta} > \underline{0}), \qquad (4)$$

and

$$h(x) = \frac{f(x)}{R(x)} = \left[\varphi(\underline{\theta})\ x^{\theta_4-1}\left[1+\frac{\theta_5}{\theta_2}\,x^{\theta_1}\right]^{-\theta_3}\right]$$

$$\times\left[1-\varphi(\underline{\theta})\sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l x^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{-1},$$

$$x > 0;\ (\underline{\theta} > \underline{0}). \qquad (5)$$

The GBurr($\underline{\theta}$) distribution is related to a wide range of well-known distributions such as the generalized beta Type II, Lomax or compound gamma, inverse Lomax, Gompertz, F-distribution, Burr Type XII, Pareto Type I, exponential, Rayleigh, beta Type I, compound Gompertz, Weibull, Dagum, inverted beta (beta Type II), gamma, compound Weibull, chi-squared, half normal, half standard normal, Log-Logistic (Fisk) and Type I generalized logistic distributions.[For more details see, Kibria and Shakil (2011) and AL-Sayed (2017)].

The Burr distributions attract special attention in life testing, reliability analysis and hypothesis testing as it is applied in several areas such as economics, forestry, exotoxicology and environmetrics among others. [For more details about the Burr distributions, see, Surles and Padgett (2005), Pushkarna *et al.* (2013), Cordeiro *et al.* (2014), Gomes *et al.* (2015), Para *et al.* (2015), Merovci *et al.* (2016), Behairy *et al.* (2016), Cordeiro *et al.* (2016) and Kim *et al.* (2016)].

Bayesian approach of Burr distributions was discussed by several authors. See, for example, Paranaíba *et al.* (2011), Paranaíba *et al.* (2012) and Ahmad *et al.* (2015). The *squared error loss* (SEL) function is the most popular symmetric loss function used in literature. The symmetric nature of SEL function gives equal weight to over and under estimation of the parameters. It takes the form

$$L(b^*, b) = c(b^* - b)^2,$$

where c denotes a constant and b^* is an estimator. The Bayes estimator under SEL function is the mean of the posterior distribution and takes the form

$$b^*_{(SE)} = E(b|\underline{x}) = \int_b b\ \pi(b|\underline{x})\,db. \qquad (6)$$

In life testing, over estimation may be more serious than under estimation or vice versa. Varian (1975) suggested the use of the *linear exponential* (LINEX) loss function; as an asymmetric loss function, to be of the form

$$L(\mathrm{b}^*,\mathrm{b}) = e^{\upsilon(\mathrm{b}^*-\mathrm{b})} - \upsilon(\mathrm{b}^*-\mathrm{b}) - 1, \qquad \upsilon \neq 0\,,$$

under LINEX loss function, the Bayes estimator $b^*_{(LINEX)}$ of b is given by

$$b^*_{(LINEX)} = \frac{-1}{\upsilon} \ln E\left(e^{-\upsilon b} \middle| \underline{x}\right), \tag{7}$$

where $E\left(e^{-\upsilon b} \middle| \underline{x}\right)$ stands for posterior expectation. [See, Zellner (1986)].

The general problem of prediction may be described as that of inferring the values of unknown observables (future observations, known as *future sample*), or functions of such variables, from current available observations, known as *informative sample*. Prediction has been applied in a variety of disciplines such as medicine, engineering, business, economic and other areas. [For more details, see Aitchison and Dunsmore (1975), AL-Hussaini (2010), AL-Hussaini and Hussein (2011), AL-Hussaini and Ateya (2012) and Sancetta (2012)].

This paper is organized as follows: In Section 2, point and credible intervals of the parameters, rf and hrf based on informative priors are considered. The two-sample Bayesian prediction is used to predict future observables from GBurr population in Section 3. In Section 4, Monte Carlo simulation study is performed to investigate the results of Bayesian estimation and prediction. Two applications are used in Section 5 to demonstrate how the proposed methods can be used in practice.

2. Bayesian Estimation

In this section, the Bayesian point and credible intervals estimation for the parameters, rf and hrf of the GBurr($\underline{\theta}$) distribution are derived.

Suppose that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ is a censored sample of size r obtained from a life-test on n items (Type II censored sample) whose lifetimes have the GBurr($\underline{\theta}$) distribution, then the *likelihood function* (LF) is given by

$$L(\underline{\theta}|\underline{x}) \propto \psi(\underline{\theta},\ \underline{x})\, \phi(\underline{\theta},\ x_{(r)}), \tag{8}$$

where

$$\psi(\underline{\theta},\ \underline{x}) = \left[\prod_{i=1}^{r}\left(\varphi(\underline{\theta})\, x_{(i)}^{\theta_4-1}\left[1+\frac{\theta_5}{\theta_2}\, x_{(i)}^{\theta_1}\right]^{-\theta_3}\right)\right]$$

and

$$\phi(\underline{\theta},\ x_{(r)}) = \left[1-\left[\varphi(\underline{\theta})\sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l x_{(r)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]\right]^{n-r}. \tag{9}$$

Assuming that the parameters $\underline{\theta}$ are unknown and independent. Then the joint prior distribution, of $\underline{\theta}$, is given by

$$\pi(\underline{\theta}) \propto \prod_{j=1}^{5}\pi(\theta_j), \qquad j = 1, 2, \ldots, 5. \tag{10}$$

Considering the prior knowledge of the vector of parameters $\underline{\theta}$, is adequately represented by conjugate prior which is the inverted beta (beta Type II) distribution with parameters α_j and β_j**.** Then the joint prior distribution of all the unknown parameters has a joint pdf given by

$$\pi\left(\underline{\theta};\, \underline{\alpha}, \underline{\beta}\right) = \prod_{j=1}^{5}\left[\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right],$$

$$\theta_j > 0;\ \left(\alpha_j, \beta_j > \underline{0}\right),\quad j = 1, 2, \dots, 5, \tag{11}$$

where α_j and β_j are the hyper- parameters of the joint prior distribution.

Combining the LF in (8) and the joint prior distribution given by (11), then the joint posterior density of the parameters $\underline{\theta}$, given $\underline{x} = (x_1, x_2, \dots, x_r)$ is given by

$$\pi\left(\underline{\theta}\middle|\underline{x}\right) = K\,\psi\left(\underline{\theta}, \underline{x}\right)\phi\left(\underline{\theta},\, x_{(r)}\right)$$
$$\times\left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right], \tag{12}$$

where

$$K^{-1} = \int_{\underline{\theta}} \psi\left(\underline{\theta},\, \underline{x}\right)\phi\left(\underline{\theta},\, x_{(r)}\right)$$
$$\times\left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta}\,, \tag{13}$$

where $\psi\left(\underline{\theta},\, \underline{x}\right)$ and $\phi\left(\underline{\theta},\, x_{(r)}\right)$ are given by (9),

$\int_{\underline{\theta}} = \int_{\theta_1} \int_{\theta_2} \int_{\theta_3} \int_{\theta_4} \int_{\theta_5}$ and

$$d\underline{\theta} = d\theta_5\, d\theta_4\, d\theta_3\, d\theta_2\, d\theta_1\,. \tag{14}$$

2.1 Point estimation

Bayes estimators are considered under two different loss functions, SEL function as a symmetric loss function and LINEX loss function as an asymmetric loss function. The Bayes estimators of the parameters, rf and hrf under SEL function and LINEX loss function can be obtained from (6), (7) and (12) as given below

$$\delta^*_{(SE)} = E\left(\delta\middle|\underline{x}\right) = \int_{\underline{\theta}} \delta\ \pi\left(\underline{\theta}\middle|\underline{x}\right) d\underline{\theta}$$

$$= \int_{\underline{\theta}} \delta\, K\,\psi\left(\underline{\theta},\, \underline{x}\right)\phi\left(\underline{\theta},\, x_{(r)}\right)$$

$$\times\left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta}, \tag{15}$$

and

$$\delta^*_{(\text{LINEX})} = \frac{-1}{\upsilon}\ln \int_{\underline{\theta}} e^{-\upsilon\delta}\ K\,\psi\left(\underline{\theta},\, \underline{x}\right)\phi\left(\underline{\theta},\, x_{(r)}\right)$$
$$\times\left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta}, \tag{16}$$

where δ is $\theta_j;\ j = 1, 2, \dots, 5$, rf or hrf, respectively.

2.2 Credible intervals

The Bayesian analog to the confidence interval is called credible interval. In general, a two-sided 100(1-τ)% credible intervals of $\underline{\theta}$, are given by

$$P\left[L_j(\underline{x}) < \theta_j < U_j(\underline{x})|\underline{x}\right] = \int_{L_j(\underline{x})}^{U_j(\underline{x})} \pi(\theta_j|\underline{x})\, d\theta_j$$
$$= 1-\tau, \qquad j=1, 2, \ldots, 5, \quad (17)$$

where $L_j(\underline{x})$ and $U_j(\underline{x})$ are the lower and upper limits.

Since, the joint posterior distribution is given by (12), then a two-sided 100(1-τ)% credible intervals of $\underline{\theta}$, as given below

$$P(\theta_\ell > L_\ell(\underline{x})|\underline{x}) = \int_{L_\ell(\underline{x})}^{\infty} \int_{\underline{\theta}_j} K\, \psi(\underline{\theta},\, \underline{x})\, \phi(\underline{\theta},\, x_{(r)})$$
$$\times\left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta}_j d\theta_\ell$$
$$= 1-\frac{\tau}{2}, \quad (18)$$

and

$$P(\theta_\ell > U_\ell(\underline{x})|\underline{x}) = \int_{U_\ell(\underline{x})}^{\infty} \int_{\underline{\theta}_j} K\, \psi(\underline{\theta},\, \underline{x})\, \phi(\underline{\theta},\, x_{(r)})$$
$$\times\left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\, \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta}_j d\theta_\ell$$
$$= \frac{\tau}{2}, \quad (19)$$

where $\ell \neq j$ and ℓ, j=1,2,…,5.

3. Bayesian Two-Sample Prediction

Considering that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ are the first r ordered life times in a random sample of n components (Type II censoring) whose failure times are identically distributed as a random variable X having the GBurr($\underline{\theta}$) distribution, informative sample, given by (1) and that $Y_{(1)}, Y_{(2)}, \ldots, Y_{(m)}$ is a second independent random sample (of size m) of future observables from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample.

For the future sample of size m, let $Y_{(s)}$ denotes the s^{th} order statistic, $1 \leq s \leq m$. The pdf of $Y_{(s)}$ is given by

$$h\left(y_{(s)}|\underline{\theta}\right) = D(s)\, \varphi(\underline{\theta})\, y_{(s)}^{\theta_4-1}\, [1+\frac{\theta_5}{\theta_2}\, y_{(s)}^{\theta_1}]^{-\theta_3}$$
$$\times\left[\varphi(\underline{\theta}) \sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{s-1}$$
$$\times\left[1-\varphi(\underline{\theta}) \sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{m-s},$$

$$y_{(s)} > 0; (\underline{\theta} > \underline{0}), \tag{20}$$

where

$D(s) = s\binom{m}{s} = \frac{m!}{(s-1)!(m-s)!} = \frac{1}{B(s,m-s+1)}$ and

$$s = 1, 2, 3, \dots, m. \tag{21}$$

The *Bayesian predictive density* (BPD) function of $Y_{(s)}$ given $\underline{x}$ is given by

$$h(y_{(s)}|\underline{x}) = \int_{\underline{\theta}} h(y_{(s)}|\underline{\theta})\, \pi(\underline{\theta}|\underline{x})\, d\underline{\theta}, \tag{22}$$

where

$\int_{\underline{\theta}}$ and $d\underline{\theta}$ are given by (14), $s = 1, 2, 3, \dots, m$,

$h(y_{(s)}|\underline{\theta})$ and $\pi(\underline{\theta}|\underline{x})$ are respectively the conditional predictive density of the s^{th} order lifetime and the joint posterior density.

The BPD of the future observation $Y_{(s)}$ given $\underline{x}$ can be obtained by substituting (12) and (20) into (22) as given below

$$h(y_{(s)}|\underline{x}) = \int_{\underline{\theta}} K\, D(s)\, \varphi(\underline{\theta})\, \psi(\underline{\theta}, \underline{x})$$
$$\times \phi(\underline{\theta}, x_{(r)})\, y_{(s)}^{\theta_4 - 1} \left[1 + \frac{\theta_5}{\theta_2} y_{(s)}^{\theta_1}\right]^{-\theta_3}$$
$$\times \left[\varphi(\underline{\theta}) \sum_{l=0}^{\infty} \left[\frac{(-1)^l \binom{\theta_3 + l - 1}{l} \left(\frac{\theta_5}{\theta_2}\right)^l\, y_{(s)}^{(\theta_4 + \theta_1 l)}}{(\theta_4 + \theta_1 l)}\right]\right]^{s-1}$$
$$\times \left[1 - \varphi(\underline{\theta}) \sum_{l=0}^{\infty} \left[\frac{(-1)^l \binom{\theta_3 + l - 1}{l} \left(\frac{\theta_5}{\theta_2}\right)^l\, y_{(s)}^{(\theta_4 + \theta_1 l)}}{(\theta_4 + \theta_1 l)}\right]\right]^{m-s}$$
$$\times \left[\prod_{j=1}^{5} \left(\frac{1}{B(\alpha_j, \beta_j)}\, \theta_j^{\alpha_j - 1} (1 + \theta_j)^{-(\alpha_j + \beta_j)}\right)\right] d\underline{\theta}, \tag{23}$$

where $\varphi(\underline{\theta})$ is given by (3), $\psi(\underline{\theta}, \underline{x})$ and $\phi(\underline{\theta}, x_{(r)})$ are given by (9), K^{-1} is given by (13) and $D(s)$ is given by (21).

3.1 Point prediction

Based on Type II censoring, Bayesian prediction is considered under two types of loss functions SEL function, as a symmetric loss function, and LINEX loss function, as an asymmetric loss function. Then, the *Bayes predictive estimator* (BPE) for the future observation $Y_{(s)}$, under SEL function is given by

$$\hat{y}_{(s)(SE)} = E(y_{(s)}|\underline{x}) = \int_{y_{(s)}} y_{(s)}\, h(y_{(s)}|\underline{x})\, dy_{(s)}$$
$$= \int_{\underline{\theta}_*} K\, D(s)\, \varphi(\underline{\theta})\, \psi(\underline{\theta}, \underline{x})$$

$$\times \phi(\underline{\theta}, x_{(r)})\ y_{(s)}^{\theta_4} \left[1+\frac{\theta_5}{\theta_2} y_{(s)}^{\theta_1}\right]^{-\theta_3}$$

$$\times \left[\varphi(\underline{\theta}) \sum_{l=0}^{\infty} \left[\frac{(-1)^l \binom{\theta_3+l-1}{l} \left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{s-1}$$

$$\times \left[1-\varphi(\underline{\theta}) \sum_{l=0}^{\infty} \left[\frac{(-1)^l \binom{\theta_3+l-1}{l} \left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{m-s}$$

$$\times \left[\prod_{j=1}^{5} \left(\frac{1}{B(\alpha_j,\beta_j)} \theta_j^{\alpha_j-1} (1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta_*}, \qquad (24)$$

and the BPE of the future observation $Y_{(s)}$, under LINEX loss function is given by

$$\hat{y}_{(s)(\mathrm{LINEX})} = \frac{-1}{v} \ln E\left(\exp(-v y_{(s)}) \,|\underline{x}\right)$$

$$= \frac{-1}{v} \ln \int_{y_{(s)}} \exp(-v y_{(s)})\ h(y_{(s)}|\underline{x})\ dy_{(s)}$$

$$= \frac{-1}{v} \ln \int_{\underline{\theta_*}} K\ \exp(-v y_{(s)})\ D(s)\, \varphi(\underline{\theta})\ y_{(s)}^{\theta_4-1}$$

$$\times \psi(\underline{\theta}, \underline{x})\ \phi(\underline{\theta}, x_{(r)}) \left[1+\frac{\theta_5}{\theta_2} y_{(s)}^{\theta_1}\right]^{-\theta_3}$$

$$\times \left[\varphi(\underline{\theta}) \sum_{l=0}^{\infty} \left[\frac{(-1)^l \binom{\theta_3+l-1}{l} \left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{s-1}$$

$$\times \left[1-\varphi(\underline{\theta}) \sum_{l=0}^{\infty} \left[\frac{(-1)^l \binom{\theta_3+l-1}{l} \left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{m-s}$$

$$\times \left[\prod_{j=1}^{5} \left(\frac{1}{B(\alpha_j,\beta_j)} \theta_j^{\alpha_j-1} (1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] d\underline{\theta_*}, \qquad (25)$$

where

$$\int_{\underline{\theta_*}} = \int_{y_{(s)}} \int_{\theta_5} \int_{\theta_4} \int_{\theta_3} \int_{\theta_2} \int_{\theta_1} \quad \text{and}$$

$$d\underline{\theta_*} = d\theta_1 d\theta_2\, d\theta_3 d\theta_4\, d\theta_5 dy_{(s)}. \qquad (26)$$

Special cases:

I. If $s = 1$, in (24, 25), one can predict the minimum observable, $Y_{(1)}$, which represents the first failure time in a future sample of size m, under SEL function and LINEX loss function.

II. If $s = m$, in (24, 25), one can predict the maximum observable, $Y_{(m)}$, which represents the largest failure time in a future sample of size m, under SEL function and LINEX loss function.

III. If $s = \frac{m+1}{2}$, in (24, 25), one can predict the median observable in odd case, $Y_{\left(\frac{m+1}{2}\right)}$, which represents the median failure time in a future sample of size m, under SEL function and LINEX loss function.

3.2 Bayesian prediction bound

A 100(1-τ)% *Bayesian prediction bounds* (BPB) for the future observation $Y_{(s)}$, such that

$P\left(L_{(s)}(\underline{x}) < Y_{(s)} < U_{(s)}(\underline{x})\middle|\underline{x}\right) = 1-\tau,$ are given, respectively, by

$$P\left(Y_{(s)} > L_{(s)}(\underline{x})\middle|\underline{x}\right) = \int_{L_{(s)}(\underline{x})}^{\infty} \int_{\underline{\theta}} K\, D(s)\, \varphi(\underline{\theta})\; y_{(s)}^{\theta_4-1}$$

$$\times\, \psi(\underline{\theta},\, \underline{x})\, \phi(\underline{\theta},\, x_{(r)}) \left[1+\frac{\theta_5}{\theta_2}\, y_{(s)}^{\theta_1}\right]^{-\theta_3}$$

$$\times \left[\varphi(\underline{\theta}) \sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{s-1}$$

$$\times \left[1-\, \varphi(\underline{\theta}) \sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l\; y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{m-s}$$

$$\times \left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\; \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] dy_{(s)}$$

$$= 1-\frac{\tau}{2}, \qquad (27)$$

and

$$P\left(Y_{(s)} > U_{(s)}(\underline{x})\middle|\underline{x}\right) = \int_{U_{(s)}(\underline{x})}^{\infty} \int_{\underline{\theta}} K\, D(s)\, \varphi(\underline{\theta})\; y_{(s)}^{\theta_4-1}$$

$$\times\, \psi(\underline{\theta}, \underline{x})\, \phi(\underline{\theta},\, x_{(r)}) \left[1+\frac{\theta_5}{\theta_2}\, y_{(s)}^{\theta_1}\right]^{-\theta_3}$$

$$\times \left[\varphi(\underline{\theta}) \sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l\; y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{s-1}$$

$$\times \left[1-\, \varphi(\underline{\theta}) \sum_{l=0}^{\infty}\left[\frac{(-1)^l \binom{\theta_3+l-1}{l}\left(\frac{\theta_5}{\theta_2}\right)^l\; y_{(s)}^{(\theta_4+\theta_1 l)}}{(\theta_4+\theta_1 l)}\right]\right]^{m-s}$$

$$\times \left[\prod_{j=1}^{5}\left(\frac{1}{B(\alpha_j,\beta_j)}\; \theta_j^{\alpha_j-1}(1+\theta_j)^{-(\alpha_j+\beta_j)}\right)\right] dy_{(s)}$$

$$= \frac{\tau}{2}, \qquad (41)$$

where $\varphi(\underline{\theta})$ is given by (3), $\psi(\underline{\theta},\, \underline{x})$ and $\phi(\underline{\theta},\, x_{(r)})$ are given by (9), K^{-1} is given by (13) and $D(s)$ is given by (21).

4. Simulation Study

This section aims to illustrate the performance of the presented Bayes estimates on the basis of generated data from the GBurr($\underline{\theta}$) distribution. Bayes averages of the parameters, rf and hrf based on Type II censoring are computed. Moreover, credible intervals of the parameters, rf and hrf are calculated. Bayes predictors (point and interval) for a future observation from the GBurr($\underline{\theta}$) distribution based on Type II censored data are computed for the two-sample case. All simulation studies are performed using R programming language.

Tables 1 and 2 show the Bayes averages of the parameters, their *estimated risks* (ERs), *relative errors* (REs) and credible intervals, where N=10000 is the number of

repetitions, (n = 30, 60, 100), are the sample sizes, in the complete sample case. For each sample size, the censoring size is 80% and ($\alpha_j = 2, 2, 4, 2, 3,\ \beta_j = 3, 2, 1, 5, 2$) are the values of the hyper parameters. Also Tables 3 and 4 present the Bayes averages, ERs and credible intervals of rf and hrf for different values of the time x_0.
Table 7 displays the Bayes predictive estimates and bounds for the future observations based on Type II censoring using two-sample prediction technique.

5. Some Applications

The main aim of this section is to demonstrate how the proposed methods can be used in practice. Two real lifetime data sets are used for this purpose. To check the validity of the fitted models, the Kolmogorov–Smirnov goodness of fit test is performed through using R programming language.

Application 1:

The first application is given by Murthy *et al.* (2004). The data refers to the time between failures for a repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Application 2:

The second application is the vinyl chloride data obtained from clean up gradient monitoring wells in mg/L; this data set was used by Bhaumik *et al.* (2009). The data are: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4 and 0.2.

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively, 0.1344 and 0.1056.

Tables 5 and 6 present the Bayes averages of the parameters, ERs, REs and credible intervals for the real data based on Type II censoring.
Table 8 displays the Bayes predictive estimates and bounds for the future observations based on Type II censoring and two-sample prediction technique.

Concluding Remarks

1. It is noticed, from Tables 1-6, that the ERs and REs for the estimates of the parameters, rf, hrf and the credible interval lengths of the parameters, rf and hrf under LINEX loss function have less values than the corresponding ERs, REs and the credible interval lengths under the SEL function.
2. It is observed, from Table 4 that the estimated value of the rf decreases when the time x_0 increases. While the estimated value of the hrf increases when the time x_0 increases.

3. The length of the first future order statistic is smaller than the length of the last future order statistic. [Tables 7 and 8].
4. It is observed that less ERs and REs are obtained for complete sample sizes than the corresponding results for censored samples. Also, results perform better when n gets larger.

General Conclusion

In this study, Bayes estimators for the parameters, rf and hrf of the GBurr($\underline{\theta}$) distribution, based on Type II censoring, are obtained. Bayesian prediction for a new observation from the GBurr($\underline{\theta}$) distribution, based on Type II censoring, are derived. The Bayesian estimation is derived under two types of loss functions. Monte Carlo simulation is used to construct the comparisons between the results for different cases. Moreover, the results are applied on real data sets.

In most cases, The Bayes averages of the parameters, rf and hrf based on Type II censoring under LINEX loss function have the smallest ERs, REs and the credible interval lengths than the corresponding ERs, REs and the credible interval lengths under the SEL function. Bayesian estimation under different type of loss function such as general entropy and precautionary loss functions for estimating the parameters of the GBurr($\underline{\theta}$) distribution would be useful as a basis for further researches.

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Table 1

Bayes average, estimated risk, relative error and 95% credible interval of the parameters, under *SEL* function based on Type II censoring

(N=10000, $\theta_1 = 2, \theta_2 = 0.4, \theta_3 = 3, \theta_4 = 4$ and $\theta_5 = 0.2$)

n	r	θ	Average	ER	RE	Lower	Upper	Length
30	80% 24	θ_1	2.0484	0.0032	0.0283	1.9873	2.0869	0.0996
		θ_2	0.3604	0.0023	0.1198	0.3134	0.4020	0.0886
		θ_3	3.0447	0.0033	0.0191	2.9899	3.0999	0.1100
		θ_4	3.0678	0.0056	0.0187	3.0188	3.1081	0.0892
		θ_5	0.1752	0.0013	0.1803	0.1284	0.2141	0.0858
	100% 30	θ_1	1.9622	0.0016	0.0200	1.9160	1.9867	0.0707
		θ_2	0.4304	0.0011	0.0829	0.3927	0.4466	0.0539
		θ_3	2.9572	0.0021	0.0153	2.9241	2.9934	0.0693
		θ_4	3.0702	0.0052	0.0180	3.0373	3.0943	0.0570
		θ_5	0.2328	0.0012	0.1732	0.1982	0.2486	0.0504
60	80% 48	θ_1	1.9555	0.0023	0.0239	1.9248	1.9840	0.0593
		θ_2	0.3791	0.0007	0.0661	0.3414	0.4029	0.0616
		θ_3	3.0509	0.0031	0.0185	2.9954	3.0729	0.0775
		θ_4	3.0171	0.0004	0.0050	2.9993	3.0363	0.0370
		θ_5	0.1702	0.0011	0.1658	0.1351	0.1976	0.0625
	100% 60	θ_1	1.9734	0.0010	0.0158	1.9467	1.9993	0.0526
		θ_2	0.3840	0.0003	0.0433	0.3669	0.4012	0.0344
		θ_3	2.9857	0.0004	0.0066	2.9642	3.0086	0.0444
		θ_4	2.9885	0.0002	0.0035	2.9775	3.0003	0.0228
		θ_5	0.1747	0.0008	0.1414	0.1548	0.1979	0.0431
100	80% 80	θ_1	2.0229	6.1504E-04	0.0122	1.9999	2.0351	0.0351
		θ_2	0.4090	1.6634E-04	0.0326	0.3923	0.4257	0.0334
		θ_3	2.9895	2.9201E-04	0.0057	2.9607	3.0074	0.0467
		θ_4	2.9846	3.4948E-04	0.0048	2.9659	3.0012	0.0352
		θ_5	0.2200	6.1648E-04	0.1245	0.1975	0.2426	0.0450
	100% 100	θ_1	2.0013	3.9470E-05	0.0032	1.9878	2.0102	0.0224
		θ_2	0.4046	3.9282E-05	0.0158	0.3966	0.4105	0.0139
		θ_3	3.0081	8.5933E-05	0.0030	3.0003	3.0140	0.0137
		θ_4	3.0103	1.3007E-04	0.0028	2.9993	3.0166	0.0172
		θ_5	0.2053	3.5408E-05	0.0274	0.1987	0.2085	0.0098

Table 2

Bayes average, estimated risk, relative error and 95% credible interval of the parameters, under *LINEX* loss function based on Type II censoring

(N=10000, $\theta_1 = 2, \theta_2 = 0.4, \theta_3 = 3, \theta_4 = 4$ and $\theta_5 = 0.2$)

n	r	θ	Average	ER	RE	Lower	Upper	Length
30	80% 24	θ_1	1.9521	0.0028	0.0264	1.9198	1.9887	0.0689
		θ_2	0.3782	0.0006	0.0612	0.3427	0.3963	0.0535
		θ_3	2.9663	0.0015	0.0129	2.9342	2.9990	0.0648
		θ_4	3.9815	0.0004	0.0050	3.9624	3.9995	0.0371
		θ_5	0.1697	0.0010	0.1581	0.1514	0.1959	0.0444
	100% 30	θ_1	1.9838	0.0005	0.0112	1.9434	2.0052	0.0618
		θ_2	0.4146	0.0004	0.0500	0.3831	0.4342	0.0511
		θ_3	3.0134	0.0003	0.0057	2.9878	3.0304	0.0426
		θ_4	4.0068	0.0001	0.0025	3.9887	4.0220	0.0334
		θ_5	0.2227	0.0006	0.1225	0.2009	0.2340	0.0331
60	80% 48	θ_1	2.0172	4.6509E-04	0.0107	1.9911	2.0365	0.0454
		θ_2	0.3939	9.6072E-05	0.0245	0.3804	0.4019	0.0215
		θ_3	2.9787	5.4340E-04	0.0077	2.9623	2.9965	0.0342
		θ_4	4.0129	2.1923E-04	0.0037	3.9970	4.0232	0.0261
		θ_5	0.2180	3.7242E-04	0.0964	0.1991	0.2274	0.0282
	100% 60	θ_1	2.0155	3.6711E-04	0.0096	1.9924	2.0336	0.0411
		θ_2	0.4077	6.9229E-05	0.0207	0.3991	0.4128	0.0137
		θ_3	3.0079	8.1130E-05	0.0030	2.9986	3.0140	0.0155
		θ_4	3.9947	7.2130E-05	0.0021	3.9805	4.0054	0.0249
		θ_5	0.2049	7.9152E-05	0.0444	0.1916	0.2149	0.0233
100	80% 80	θ_1	1.9884	1.4764E-04	0.0061	1.9829	1.9968	0.0139
		θ_2	0.3947	4.7416E-05	0.0177	0.3843	0.4002	0.0159
		θ_3	3.0089	9.8012E-05	0.0033	2.9991	3.0155	0.0163
		θ_4	3.9941	7.8103E-05	0.0022	3.9831	4.0026	0.0195
		θ_5	0.1900	1.2265E-04	0.0500	0.1818	0.2001	0.0183
	100% 100	θ_1	1.9957	2.3109E-05	0.0022	1.9914	1.9991	0.0077
		θ_2	0.3964	1.5306E-05	0.0097	0.3926	0.3994	0.0067
		θ_3	2.9960	1.9949E-05	0.0015	2.9926	2.9996	0.0070
		θ_4	4.0021	1.0279E-05	0.0008	3.9974	4.0058	0.0084
		θ_5	0.1976	1.2588E-05	0.0173	0.1912	0.2010	0.0097

Table 3

Bayes average, estimated risk and 95% credible interval of the reliability and hazard rate functions at $x_0 = 0.1$, based on Type II censoring

n	r	Loss function	rf and hrf	ER	Average	Lower	Upper	Length
30	80% 24	SEL	$R(x_0)$	0.0538	0.7660	0.7223	0.7992	0.0768
			$h(x_0)$	0.2862	0.6057	0.5645	0.6436	0.0791
		LINEX	$R(x_0)$	0.0428	0.7903	0.7761	0.8002	0.0241
			$h(x_0)$	0.2657	0.5864	0.5578	0.6087	0.0508
	100% 30	SEL	$R(x_0)$	0.0444	0.7870	0.7473	0.8118	0.0645
			$h(x_0)$	0.2543	0.5751	0.5296	0.6006	0.0710
		LINEX	$R(x_0)$	0.0369	0.8049	0.7955	0.8118	0.0163
			$h(x_0)$	0.2515	0.5726	0.5475	0.5982	0.0507
60	80% 48	SEL	$R(x_0)$	0.0524	0.7687	0.7357	0.7994	0.0637
			$h(x_0)$	0.2849	0.6048	0.5774	0.6189	0.0415
		LINEX	$R(x_0)$	0.0404	0.7959	0.7902	0.8007	0.0105
			$h(x_0)$	0.2642	0.5853	0.5799	0.5911	0.0113
	100% 60	SEL	$R(x_0)$	0.0364	0.8064	0.7898	0.8230	0.0332
			$h(x_0)$	0.2322	0.5531	0.5358	0.5662	0.0304
		LINEX	$R(x_0)$	0.0342	0.8121	0.8086	0.8141	0.0055
			$h(x_0)$	0.2398	0.5609	0.5571	0.5641	0.0070
100	80% 80	SEL	$R(x_0)$	0.0411	0.7944	0.7840	0.8069	0.0229
			$h(x_0)$	0.2786	0.5991	0.5936	0.6034	0.0098
		LINEX	$R(x_0)$	0.0343	0.8118	0.8068	0.8164	0.0096
			$h(x_0)$	0.2612	0.5823	0.5779	0.5849	0.0070
	100% 100	SEL	$R(x_0)$	0.0354	0.8089	0.8045	0.8118	0.0072
			$h(x_0)$	0.2307	0.5515	0.5461	0.5553	0.0092
		LINEX	$R(x_0)$	0.0342	0.8122	0.8096	0.8141	0.0045
			$h(x_0)$	0.2201	0.5404	0.5367	0.5423	0.0056

Table 4

Bayes average of the reliability and hazard rate functions at different time x_0, based on Type II censoring

n	r	x_0	SEL		LINEX	
			$\hat{R}(x_0)$	$\hat{h}(x_0)$	$\hat{R}(x_0)$	$\hat{h}(x_0)$
	80%	0.1	0.7660	0.6057	0.7903	0.5864
	24	0.3	0.7479	0.6188	0.7801	0.5889
30	100%	0.1	0.7870	0.5751	0.8049	0.5726
	30	0.3	0.7774	0.5924	0.7906	0.5889
	80%	0.1	0.7687	0.6048	0.7959	0.5853
60	48	0.3	0.7610	0.6230	0.7885	0.6000
	100%	0.1	0.8064	0.5531	0.8121	0.5609
	60	0.3	0.8058	0.5629	0.8076	0.5733
	80%	0.1	0.7944	0.5991	0.8118	0.5823
100	80	0.3	0.7849	0.5999	0.8113	0.5878
	100%	0.1	0.8089	0.5515	0.8122	0.5404
	100	0.3	0.7862	0.5709	0.8019	0.5626

Table ٥

Bayes average, estimated risk, relative error and 95% credible interval of the parameters, under *SEL* function for the real data sets based on Type II censoring

	n	r	θ	Average	ER	RE	Lower	Upper	Length
Application I	30	80% 24	θ_1	2.9745	0.0007	0.0088	2.9495	2.9931	0.0436
			θ_2	0.0626	0.0025	0.5000	0	0.1013	0.1250
			θ_3	3.0202	0.0012	0.0115	2.9730	3.0701	0.0971
			θ_4	2.9804	0.0008	0.0071	2.9233	3.0160	0.0927
			θ_5	0.4942	0.0007	0.0529	0.4418	0.5327	0.0909
		100% 30	θ_1	3.0013	8.6655E-05	0.0032	2.9808	3.0149	0.0340
			θ_2	0.1111	1.7102E-04	0.1414	0.0929	0.1203	0.0274
			θ_3	2.9877	2.6013E-04	0.0058	2.9679	3.0057	0.0378
			θ_4	3.0197	5.0967E-04	0.0056	2.9947	3.0339	0.0392
			θ_5	0.5036	7.2331E-05	0.0167	0.4836	0.5156	0.0321
Application II	34	80% 27	θ_1	4.0568	0.0031	0.0136	3.9841	4.1022	0.1181
			θ_2	0.0824	0.0007	0.2646	0.0439	0.1169	0.0730
			θ_3	3.2185	0.0008	0.0088	3.1858	3.2593	0.0735
			θ_4	2.9837	0.0469	0.0433	2.9545	3.0054	0.0509
			θ_5	0.6333	0.0016	0.0667	0.5885	0.6691	0.0806
		100% 34	θ_1	4.1075	0.0002	0.0034	4.0809	4.1235	0.0426
			θ_2	0.1131	0.0004	0.2000	0.0811	0.1334	0.0523
			θ_3	3.1896	0.0004	0.0062	3.1557	3.2233	0.0675
			θ_4	2.9997	0.0403	0.0401	2.9752	3.0199	0.0446
			θ_5	0.5945	0.0001	0.0167	0.5725	0.6098	0.0372

Table 6

Bayes average, estimated risk, relative error and 95% credible interval of the parameters, under *LINEX* loss function for the real data sets based on Type II censoring

	n	r	θ	Average	ER	RE	Lower	Upper	Length
Application I	30	80% 24	θ_1	3.0036	0.0003	0.0058	2.9797	3.0309	0.0513
			θ_2	0.1115	0.0002	0.1414	0.0923	0.1228	0.0305
			θ_3	2.9935	0.0001	0.0033	2.9636	3.0071	0.0435
			θ_4	3.9768	0.0007	0.0066	3.9507	4.0010	0.0503
			θ_5	0.5165	0.0004	0.0400	0.4969	0.5327	0.0358
		100% 30	θ_1	3.0014	1.3260E-05	0.0010	2.9942	3.0068	0.0126
			θ_2	0.1065	5.0387E-05	0.0707	0.0996	0.1110	0.0114
			θ_3	2.9987	7.3098E-06	0.0009	2.9926	3.0032	0.0105
			θ_4	4.0035	1.7995E-05	0.0011	3.9973	4.0067	0.0094
			θ_5	0.4954	2.7663E-05	0.0109	0.4889	0.4990	0.0101
Application II	34	80% 27	θ_1	4.0820	5.0324E-04	0.0054	4.0558	4.1050	0.0492
			θ_2	0.0950	9.3918E-05	0.0949	0.0782	0.1085	0.0303
			θ_3	3.2151	3.0950E-04	0.0054	3.1931	3.2283	0.0352
			θ_4	4.9918	2.0032E-04	0.0028	4.9700	5.0097	0.0397
			θ_5	0.6111	3.6899E-04	0.0333	0.5762	0.6323	0.0561
		100% 34	θ_1	4.1089	9.910Ee-05	0.0024	4.0994	4.1167	0.0173
			θ_2	0.1081	8.7768E-05	0.0948	0.0968	0.1154	0.0186
			θ_3	3.2015	6.0722E-05	0.0024	3.1844	3.2149	0.0305
			θ_4	5.0019	3.1109E-05	0.0011	4.9906	5.0118	0.0212
			θ_5	0.6012	3.8731E-05	0.0105	0.5890	0.6155	0.0265

Table 7

Bayes predictive and bounds of the future observation based on Type II censoring under two-sample prediction

(N=10000, $n = 100, r = 80, m = 21, \theta_1 = 2,$ $\theta_2 = 0.4, \theta_3 = 3, \theta_4 = 4$ and $\theta_5 = 0.2$)

s	SE				LINEX (v = -.5)			
	$\hat{y}_{(s)(SE)}$	LL	UL	Length	$\hat{y}_{(s)(\mathrm{LNX})}$	LL	UL	Length
1	0.3009	0.2995	0.3024	0.0029	0.3001	0.2993	0.3008	0.0014
11	0.5010	0.4985	0.5037	0.0053	0.5008	0.4997	0.5029	0.0032
21	0.7037	0.6990	0.7061	0.0071	0.6971	0.6927	0.6992	0.0064

Table 8

Bayes predictive and bounds of the future observation for real data based on Type II censoring under two-sample prediction

$(\mathbf{m_1} = \mathbf{25} \text{ and } \mathbf{m_2} = \mathbf{35})$

Real data	*s*	SE				LINEX (v = -.5)			
		$\hat{y}_{(s)(SE)}$	LL	UL	Length	$\hat{y}_{(s)(\mathrm{LNX})}$	LL	UL	Length
Application I	1	0.1210	0.1199	0.1224	0.0025	0.1204	0.1195	0.1209	0.0014
	13	1.1779	1.1732	1.1803	0.0070	1.1785	1.1759	1.1801	0.0041
	25	2.5015	2.4967	2.5050	0.0083	2.4996	2.4948	2.5027	0.0079
Application II	1	0.0984	0.0964	0.0997	0.0033	0.0990	0.0982	0.0999	0.0016
	18	1.4991	1.4962	1.5024	0.0063	1.5020	1.4997	1.5032	0.0035
	35	8.0023	7.9975	8.0050	0.0075	8.0023	7.9977	8.0046	0.0069

Export Financing for Small and Medium Enterprises (SMEs)

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1. **Introduction:**

In general, products which are sold in domestic markets face fewer obstacles and barriers than the products for export markets. Success in exporting is more difficult than selling to the domestic market as result of the differences in culture, income level, transportation and sever competition. The ability to sell goods abroad no longer depends solely on quality, delivery and price; a significant factor of growing importance is the ability and willingness to grant credit.

The most important problems of Egyptian exports are insurance on export goods against commercial and non-commercial risks, fund shortage for exports and high cost of export financing.

Default risk insurance is the most essential factors to encourage exports. So, Egypt issued new law (no.21) in 1992 to implement Export Credit Guarantee Company of Egypt (ECGE) that aimed to promote and development Egyptian exports for industrial, commercial and agricultural sector through export guarantee processes against commercial and non-commercial risks with Export Development Bank of Egypt assistance. They insure Egyptian export against nonpayment due to commercial risks (buyer default and Bankruptcy) and political risks (country default and non-transfer). ECGE has established relations with networks of debt collectors and bank representatives to monitor and recover overdue accounts giving exporters new levels of efficiency and control with overseas buyers through avoiding losses caused by non-payment (Thuraya, 2017).

Export financing is a huge driver for economic development and helps maintain the flow of credit to exporters. In this environment of relatively low liquidity, the cost of financing has increased and suppliers especially small and medium enterprises (SMEs) are finding it more difficult to obtain the credit they need. The scarcity of cheap external financing has driven many firms to look across their financial supply chain[1] for opportunities to improve the management of working capital, optimize their cash flows and unlock trapped cash (Lekkakos, 2016.368).

The researcher has interests on export financing for Small and Medium Enterprises (SMEs) for two reasons:

- A challenge for many small business is access to financing. In particular, many firms find it difficult to finance their production cycle, since after goods are delivered. Most buyers demand 30 to 90 days to pay. For this duration, sellers issue an invoice and record as an account receivable, which is an illiquid asset until payment is received.
- Exports have to be increased to play its role in the realization of the development plan. Since this cannot be realized by the traditional exports (timber, gold and other minerals) due to sectoral and market constraints. Export diversification, with a focus on the non-traditional sector, is seen as innovative strategy for export growth.

[1] Supply chain is a linkage of operations that provide goods and services from the supplier of raw material to the end customer.

Exports also face difficulties in obtaining great amount of export financing mostly due to the risk of export transactions. Even when export risk have been assessed and payment terms properly arranged, there are still exceptional problems which can significantly delay payment and sometime cause insolvency, there are: shipping delays, a change in economic circumstances by the time the goods arrive, bills of exchange[2] not accepted, deliberate default of funds by banks, shortage of hard currency at the central bank, actual of near insolvency of the buyer (Al-Araj, 2003, pp 92-93).

Trade financing is not just about funding export transaction. It is also about limiting the risks of such transaction. Most banks need to be assured about the ability of borrowers to repay a loan before agreeing to finance export transactions. Banks thus insist on adequate collateral. Insurance policies and guarantees granted by export credit agencies can be used as collateral for trade financing. Since banks are often willing to grant exporters favorable credit conditions once the perceived risk of default has been reduced.

2. Export Credit Agencies:

Most firms involved in inter-firm trade offer credit to their customers, where trade credit is defined as allowing customers to obtain goods or services and pay at a later date. Trade credit is an important source of short-term finance for business and represents a substantial component of both corporate liabilities and assets (Summers, 2000, 37).

[2] Document signed by the person authorizing, which tells another to pay money unconditionally to a named person on a certain date.

Export Credit Agencies (ECA) play a role in international trade and investment flows. ECA is a vital part of the infrastructure supporting trade and was often considered to be a critical component in a nation's export-led growth strategy. The following consideration are strong reasons for establishing an ECA: (i) protection against risks, (ii) access to bank financing, (iii) access to information, (iv) an instrument of government policy and international Co-operation.

The classic way of laying off any risk is to take insurance, and international credit risks are no exception. The main general principles of export credit insurance may be summarized as follows:

a. Both insured and insurer must share the risk. Therefore credit insurance policies usually cover between 75 and 95 percent of the loss according to the type of policy.

b. Cover is restricted to agree debts and disputes must be resolved before a claim under the policy can be paid.

c. The terms of payment must be appropriate to the goods or services concerned.

d. Credit insurance does not guarantee payment at due and is not a financial guarantee.

In order to meet the World Trade Organization (WTO) requirement, it is clear that, on the long run, premiums should match the risks. Although the above mentioned coverage ratios are still in the same range, the risk sharing in general between

ECAs and the private market can be developed further, in order to create more capacity for exporters (Al-Araj, 2003, p.25-26).

Credit risk is pervasive throughout financial markets. Traditionally, various financial institutions have assumed the burden of credit risk. Banks bear the credit risk attached to bank loans. Credit insurance companies have provided coverage for commercial credit risk faced by suppliers of consumer and investment goods and services. Public insurers, such as the ECGD in the UK have specialized in the coverage of credit risk attached to export trade and overseas investment. Specialized institutions, such as factoring companies, have offered credit risk coverage as one component in a basket of financial service. More recently, the increase of financial contract that involve counter-party default risk such as swaps, back-to-back loans and derivative products have focused attention on ways to deal with credit risk in the marketplace. Products such as credit default swaps, credit spread options and total rate of return swaps have allowed firms and financial institutions to more effectively deal with credit risks (Loubergé, 2005, pp 118-119). Given good management, access to credit will accelerate the adoption of more efficient and effective techniques of production and marketing by exporters (Buatsi,2002,504).

Export financing is often a key factor in a successful sale. Contract negotiation is important, but at the end of the day, a company must get paid. Exporters naturally want to get paid as possible, while importers usually prefer to delay payment until they have received or resold the goods. Because of the intense competition for export markets, being able to offer attractive payment terms customary in the trade is often necessary to make a sale. Hence, exporters should be aware of various financing options open to them so that they choose the most acceptable one to both the buyer and the seller in order to reduce cost and minimize interest rate (Al-Araj, 2003, p.19).

3. Export financing

The financial needs of exporters may be linked to two distinct phases of the export process; the pre-shipment phase and post-shipment process (Buatsi, 2002, 503-504).

- Pre-shipment finance relates to funds needed by an exporter to produce or buy goods for export. It is the finance to provide the working capital between the time of receipt of an export order and the time of shipment. The need for external financing by firms may begin at the pre-shipment stage, when funding for the purchase of inputs (whether these be raw materials or capital equipment and spares), processing and other operations. Exporters require funding for a wide range of inputs and activities to purchase and/or produce goods, tools and machinery, processing, packaging, marketing. This type of finance is particularly important for small firms that have limited access to long-term capital markets and therefore, need to rely on trade credit and short-term loans. The funding of working capital will ensure continuous operations of their export business. Exporters could obtain finance at the pre-shipment stage through anticipatory letter of credit and banking credits that include short-term loans.

- Post-shipment finance is needed to bridge the gap between the time of shipment and receipt of export proceeds. Exporters usually have to wait for some time (float time) before payment is received from overseas buyers. The period of waiting depends on the terms of payments, so, the need for post-shipment finance to strengthen the financial position of the exporters varies accordingly. Based on different terms of payments, banks have devised various methods of financing through negotiations of bills of exchange. The firm's ability to complete effectively and to win contracts depends on their capacity to offer attractive credit terms to foreign buyers. The longer the credit term extended to foreign buyers, the bigger the strain on the exporters' liquidity and the more important their access to adequate external trade financing during the post-shipment stage. Exports should be aware of two factors that have to be considered in making decisions about export financing; first is need for financing to make the export process in favorable and competitive payment terms. Second, should be a consideration for the life of product financed since exporters wait long time before receiving payments (Al-Araj, 2003, p.25).

A sound financial system providing adequate credit and insurance facilities is therefore essential for exporters in developing and transition economics (Buatsi, 2002, 504). Banks, in export financing use the same principles of good lending to extend export credit facilities, they also have to make it clear on how the bank would expect repayment in due course of time. Many lending decisions are based on experience that comes from learning and applying principles of good lending. Normally, banks follow the well-known mnemonic CAMPARI, which stands for: character of the customer, ability to borrow and repay, margin of profit, purpose of the loan, amount of the loan, repayment terms and insurance against the possibility of non- payment (Al-Araj, 2003, p.22-23).

Solid financial base for an exporter is a necessary ingredient for expansion in activities and growth of the earnings. Without finance or credit, non-traditional exporters have little chance of increasing production.

Export financing is the provision of credit and any form of financial assistance to meet the needs of an exporter in carrying out an export order. Basically, there are four methods of financing an export shipment. The methods to adopt include (Buatsi, 2002, 503), (Al-Araj, 2003, p.19-20):

- Cash –in-advance
- Open account
- Documentary bills of exchange
- Commercial letters of credit

Cash-in-advance is unlikely to happen and in which case there would be no credit to manage.

Open account is the riskiest method of settlement since goods are shipped in advance of payment. The buyer is under no pressure to pay other than through fear of losing the seller's goodwill or eventually of some form of legal action (Al-Araj, 2003, p.19). Completing a transaction with international buyers carries a certain amount of payment uncertainty or delay which can be lowered with domestic buyer, thus international trade contains some degree of credit risk, exposing the exporters to danger. In particular, exporters may be unware or lack the information of the buyer's financial situation. There is also the difficulty of trying to collect an overdue account from an international buyer (Han, 2016, p.105-106). With the increase of traders in international transactions and familiarization with their trading partner's situation, the need for risk hedging is decreasing. The change in behavior reflect the growing importance of open account transactions, whereas the exporter delivers goods and the importer pays on reception or under agreed payment condition. Open account take about 80 percent of trade transactions by volume (Han, 2016, 0.106).

A documentary bills of exchange essentially involves the use of a third party, almost invariably a collecting bank, to act as an intermediary, and a fee to exchange documents of sale for payment or a promise to pay.

Commercial letters of credit is an old method of settlement likely to become outdated as the world modernized and international trade moved towards better technologies. For the best possible security, the right form of credit must be used and the documentation presented against it must be 100 percent accurate and delivered on time.

4. Factoring:

Other types of supply chain financing have been developed such as factoring that involves a process where a specialized firm takes the responsibility of collection and administration of account receivables to its customers. It can be considered as a short-term funding based on the sale of account receivables on the basis of an interest rate for service performed (Cela et al. 2013, pp112). Other defined it as a form of asset-based finance based on the value of the borrower's accounts receivables which are sold at discount to the factor company and considered the primary source of repayment (Klapper, 2006). Briefly, factoring is a financing technique that refers to the sale of the firm's accounts receivables to a financial institution known as a factor. The firm and the factor agree on the basic credit terms for each customer. The customer sends payment directly to the factor who bears the risk of default customers. The factor buys the receivables at a discount of the value of the invoice amount.

Factoring is used in developed and developing countries around the world. In 2004, total worldwide factoring volume was over US$ 860 billion, an impressive growth rate of 88% since 1998. However, the role of export factoring in both developed and developing countries is relatively small; less than 10% of factoring in developing countries is international versus about 20% in developed countries. One reasons is that exporters often rely on other products to facilitate foreign sales, such as foreign credit insurance and letters of credit (Klapper, 2006).

Factoring contracts do not involve a credit relationship but rather the transaction contains a sale and a purchase (Vazquez, 2016, 4). Factoring would be a type of discounting without recourse, because it is the sale of export account receivables to a third party which takes the credit risk while a factor may be a factoring house or a special department in a commercial bank. Under the export factoring arrangement, the seller passes its order to the factor for approval on the credit risk. Once the order has been approved, the exporter has complete protection against bad debts and political risk. The customer pays the factor which acts as the exporter's credit and collection department. The period of settlement generally does not exceed 180 days (Al-Araj, 2003, p.23-24). Problems can arise when there are assignment limitations: for instance, some countries like South Korea do not permit assignment of claims (Han, 2016, p.106). Factoring provides to the supplier collection of 70 to 90 percent of the invoice amount within 24 or 48 hours once the factoring agreement is set. When factoring company receives the payment of business bills, to difference of 10 to 30 percent of the amount is deducted the commission of the factoring company which is 1,5 to 3 percent of the invoice amount per month and then the rest is paid to the business. Factoring will not fund bad loans to protect both parties, factor and the supplier (Cela et al. 2013, pp112).

Organizations may choose to manage the integrate credit administration process 'in-house' (vertically integrate), in contrast some or all of these activities can be 'out-sourced' to specialized institutions that perform administration functions such as factors, invoice discounters, credit reference agencies, in particular a firm's requirements for short term financing. Factoring relationships involve complex bilateral contracts between the firm and the factor (summers, 2000, pp38). We have to considered that involvement of a third party such as a factor, in credit management, could leave the firm with less flexibility in the way it sets and varies credit terms between customers for price discrimination or other competitive reasons (summers, 2000, pp42).

Factoring can be either on a "non-recourse" or "recourse" basis against the factor's client (the sellers). In non-recourse factoring, the lender not only assume title to the accounts, but also assumes most of the default risk because the factor does not have recourse against the supplier if the accounts default. Factoring can also be done on either a notification or non- notification basis. Notification means that the buyers are notified that their accounts have been sold to a factor (Klapper, 2006).

5. **Benefits of factoring:**

- Factoring can be viewed as a bundle of activities. In addition to the financing component, factors typically provide two other complementary services to their clients; credit services and collection services. Credit services involve assessing the creditworthiness of the seller's customers whose accounts the factor will purchase (Klapper, 2006). Use of factors can give rise to savings in information costs for the customers of firms which use the factoring service. The factors can be a source of information to potential buyers on the price, quality and other attributes of the firm's goods in a sales situation. In any market with imperfect information there are costs to both buyer and seller in acquiring enough information to evaluate the likely risk/return ratio on a transaction. In some circumstances a factor may be able to obtain information on the credit worthiness of a buyer and to monitor buyer risk more cheaply than a supplier could, such circumstances might therefore be expected to increase the propensity of the supplier to use factoring services (Summers, 2000, pp 39). In addition to allow SMEs to effectively outsource their credit and collection functions. These credit and collection services are often especially important for receivables from buyers located overseas. Export factoring can facilitate and reduce the risk of international sales by collecting foreign account receivables. The seller's factor will typically contact a factor in the buyer's home country who will do a credit check on the buyer. The approval of the factor arrangement send an important signal to the seller before entering a business relationship (Klapper, 2006).

- There are two main differences between factoring and bank loan in favor of factoring: the first, factoring uses the account receivables which are assets not yet completed as collateral for financing. This is the benefit of the new companies which need working capital but have not enough assets to use as collateral for bank loans. The second difference lies in the assessment of credit risk, while banks are interested in credit analysis of the firm seeking credit (suppliers) and look to collateral only as a secondary source of payment. In the case of factoring, the seller's viability and creditworthiness, though not irrelevant, are only of secondary underwriting importance (Cela et al. 2013, pp110).
- It is well suited for financing receivables from high-risk seller whose receivables are obligation of buyers who are more creditworthy than the seller itself. The study of (Summers, 2000) provides more evidence that the smaller companies have a higher probability of using factoring because the

problems which small firms have in raising institutional finance and also study of (Cela et al. 2013) indicates that the factoring is a financing source more convenient for Albanian SMEs that need liquidity and working capital especially new enterprises, which have 1-5 years of operation in the market.

- Financing through factoring is one of the tools that require less time and one of the easiest method for providing liquidity for a company that have a positive result in the income statement (Cela et al. 2013).
- It is suitable for week business environment and weaker protection of creditor rights since the factored receivables are removed from the bankruptcy estate of the seller and become the property of the factor. A key legal issue is whether a financial system's commercial law recognizes factoring as a sale and purchase. If it does, then creditor rights and enforcement of loan contracts diminish in importance for factoring because factors are not creditors. That is, if a firm becomes bankrupt, its factored receivables would not to be of the bankruptcy state because they are the property of the factor, not the property of the bankrupt firm. (Vazquez, 2016), but (Klapper, 2006) finds week evidence that factoring is relatively larger in countries with weak contract enforcement.

However, factoring may still be hampered by weak contract enforcement institutions other tax, legal and regulatory impediment. Weaker governance structure may also create additional barriers to the collection of receivables in developing countries. Tax treatment of factoring transaction often makes factoring prohibitively expensive. For example, some countries that allow interest payment to banks to be tax deductible do not apply the same deduction to the interest on factoring arrangement. In addition, capital control may prevent non-banks from holding foreign currency accounts for cross-border assignments. Weak information infrastructure may also be problematic for factors. The general lack of data on payment performance, such as the kind of information that is collected by public or private credit bureaus or by factors themselves, can discourage factoring (Klapper, 2006).

One solution to these barriers to factoring is “Reverse Factoring”. In this case, the lender purchases account receivables only from specific informationally transparent, high quality buyers. The factor only needs to collect credit information and calculate the credit risk for selected buyers, such as large, internationally accredited firms. The main advantage of reverse factoring is that the credit risk is equal to the default risk of the high-quality customer ant not the risky SME (Klapper, 2006).

6. Literature review:

It may be that firms are attracted to factoring for other reasons (such as unsatisfied demand for finance) but their ability to collect information on creditworthiness, and hence be an information source, makes them more attractive to the factor. It may be that these firms can negotiate a better deal with the factor based on the information they collect. He provides more evidence that the smaller companies have a higher probability of using factoring (Summers, 2000, 47).

(Summers, 2000) investigates the decision by a firm to externalize the majority of the credit management function and use factoring. Data of this study is from 655 firms which responded to a mail survey in 1994. The study provides more evidence that the smaller companies have a higher probability of using factoring because the problems which small firms have in raising institutional finance.

(Buatsi, 2002) study was undertaken on a sample of non-traditional exporting firms and selected banks in Ghana. Ghanaian exporters hardly obtain finance for export operations. SME exporters hardly meet the requirements of banks to access credit, interest rates are high and default on loans has been high.

(Al-Araj, 2003) the purpose of this study is to develop a strategy for financing industrial exports by banks in Jordan. Sources of data are divided into primary and secondary; primary data was collected through a survey questionnaire distributed to 17 banks and 32 industrial exporting firms. Secondary data was collected from the annual and monthly reports of the central bank and export firm from 1986-2001 in the regression equation. The results indicate that industrial exports do not have sufficient credit facilities to finance their exports. The bankers of Jordan should start to use the new instruments of export financing such as factoring, forfaiting and invoice discounting in addition to improving the export credit insurance schemes in order to solve the problem of unwillingness to lend against foreign receivables.

(Loubergé, 2005 et. al) develop a method for coping with credit risk by decomposing this risk into idiosyncratic and systematic components that may be treated separately and show how this decomposition redesign financial contracts.

(Cela et al. 2013) examined factoring as the short term for SME in Albania. From statistical analysis and interviews with specialists of four factoring companies and 110 enterprises use bank loan or factoring. The results indicates:

- The factoring is a financing source more convenient for Albanian SMEs that need liquidity and working capital especially new enterprises, which have 1-5 years of operation in the market.
- The first companies that started to operate in factoring market encounter difficulties which relates to lack of information of the suppliers to providing it to clients.

(Vazquez, 2016) using a sample of 4348 firms from 25 European countries and analyzing whether the use of factoring by SMEs differs across countries due to differences in the legal protection of creditors. Results shows that the likelihood of using factoring increases with weaker protection of creditor rights and the size of factoring industry is larger in countries with greater economic development and higher rates of growth.

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MEASURING THE EFFICIENCY OF INDUSTRIES BY FUZZY DATA ENVELOPMENT ANALYSIS

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MEASURING THE EFFICIENCY OF INDUSTRIES BY FUZZY DATA ENVELOPMENT ANALYSIS

***Abstract*—** Manufacturing is a part of the income of any country, helping to grow the economy by generating productivity, stimulating research and development, and investing in the future. Therefore, this paper seeks to explain the productivity growth performance of Ethiopian's manufacturing sector using a dataset of 14 types of industries for the year of 2008; utilizing data envelopment analysis (DEA) techniques either traditional or fuzzy DEA models. Data envelopment analysis (DEA) is a methodology for measuring the relative efficiencies of a set of decision making units (DMUs) that use multiple inputs to produce multiple outputs. Conventional DEA models assume that input and output values should be certain (crisp data). However, the observed values of the input and output data in real-world situations are sometimes vague or imprecise. In this paper, three approaches that transform the original data (crisp data) into interval data, in the form of upper and lower frontier data, are suggested. Then, by using these upper and lower frontier data; the interval DEA efficiency scores can be achieved. These approaches are applied on the real-life data and the results show that data envelopment analysis (DEA) techniques are suitable to evaluate and compare the performances of industries that enable the decision makers to analyze the situation better.

***Keywords* —** Data envelopment analysis; Fuzzy; Interval data; Efficiency; Decision making units; Manufacturing industries.

I. INTRODUCTION

The manufacturing sector is considered the backbone of development in general and economic development in particular. Moreover, the economic strength of any country is measured by the development of manufacturing industries. Industries help in eradicating unemployment and poverty, reducing the heavy dependence of people on agricultural income, bringing down regional disparities by establishing industries in tribal and backward areas an exporting of manufactured goods that expand trade and commerce and bring in much needed foreign exchange. Therefore, this paper seeks to explain the productivity growth performance of Ethiopian's manufacturing sector using a dataset of 14 types of industries for the year of 2008; utilizing data envelopment analysis (DEA) techniques either traditional or fuzzy DEA models.

Data envelopment analysis (DEA) is a non-parametric technique for evaluating and measuring the relative efficiency of decision making units (DMUs) characterized by multiple inputs and multiple outputs. DEA is a linear programming technique that computes a comparative ratio of weighted outputs to weighted inputs for each unit, which is reported as the relative efficiency score. The efficiency score is usually expressed as either a number between zero and one (0-1) or as a percentage (0-100%). A decision-making unit with a score equal one becomes the efficient unit. On the other hand, a unit with a score less than one is deemed inefficient relative to other efficient units [1, 2]. The name of DEA was due to constructing an efficient frontier from efficient units by the model that this frontier will cover (envelope) the inefficient units [3]. DEA has initially been used to investigate the relative efficiency of non-for-profit organizations and it is quickly spread to profit-making organizations. DEA has been successfully applied in such diverse settings as schools, universities, hospitals, libraries, banks, shops, industries, and more recently, whole economic and society systems; in which outputs and inputs are always multiple [4, 5]. This paper is organized as follows: The next section contains literature review about using DEA in industry field. Section 3 presents a discussion about conventional models of DEA and some models of FDEA and development for these some FDEA to express the data as interval data. In section 4, an application based on a real data is presented. Section 5 represents final results and conclusion.

II. LITERATURE REVIEW

In this section we mention some of the studies that have been carried out to examine the technical efficiency using DEA models for various industries in different countries.

The study of Saricam, C. and Erdumlu, N. (2012) provided a framework for DEA application in determination and comparison of efficiency performance in an industry level. In this study, the performances of the companies in the textile, apparel and leather industry quoted by Istanbul Stock Exchange for the period 2003-2008 were analyzed by input-oriented model under variable returns to scale assumption. Net assets and the average number of employees were used as inputs, and gross value added, profit before tax and export revenues were used as the outputs. The input and output variables were selected by considering the fact that inputs and outputs should be independent in DEA in order to obtain reasonable results. The results showed that DEA is a suitable tool to make performance evaluation and to compare the performances of industries enabling the decision makers to better analyze the situation [6].

Ahmadi, V. and Ahmadi, A. (2012) examined the technical efficiency level of manufacturing industries in Iran during 2005 to 2007 (The Fourth Development Program) by using two models, CCR and BCC, based on output orientated. Number of employees, capital formation (million rials), raw materials employed by manufacturing industries (million rials), fuel employed by manufacturing industries (million rials) are used as input variables. Whereas value added of manufacturing industries (million rials) is used as output variable. According to the results, it is found that there are three principal manufacturing industries and two provinces which are identified as the best performers, namely tobacco, transport equipment and coal coke. Among thirty provinces, Bushehr and North Khorasan provinces have the best performance [7].

The study of Kumar, S. and Arora, N. (2012) involved the realization of two principal objectives. The first objective is to analyze the inter-temporal and inter-state variations in the technical efficiency of Indian sugar industry using the longitudinal data for 12 states over the period of 31 years (i.e., from 1974/75 to 2004/05). This has been accomplished by using the method of full cumulative data envelopment analysis (FCDEA). Another principal objective of this study is to identify the determinants of technical efficiency in Indian sugar industry for which the panel data Tobit regression has been used. The results suggest that the extent of technical inefficiency in Indian sugar industry is about 35.5 percent per annum, and the availability of skilled labor and profitability have been found to be most significant determinants of technical efficiency in Indian sugar industry [8].

Chueh, H. and Jheng, J. (2012) used a two-stage input-oriented BBC model to analyze the operational efficiency and profitability of the Taiwanese solar power industry between 2010 and 2011. According to the results of this study, only Motech, in 2010, was simultaneously operationally and profitably efficient. As the solar cell market expands, various input and output factors may be considered. This study constructed a performance evaluation model by using data envelopment analysis for the solar cell industry to assist relevant manufacturers in the Taiwanese solar power industry in formulating operational strategies; guidelines on future development in the industry have been recommended [9].

Saraçli et al. (2013) examined the efficiencies of 64 marble factories in Afyonkarahisar city, the most famous city in terms of marble production in Turkey. In this study, data envelopment analysis (DEA) was used to determine the efficient and non-efficient factories. The study used 12 Input and 2 output variables to examine of efficiencies of the factories according to the model of production. The inputs include: number of factory workers, number of engineers employed at the factory, number of machines in the factory, number of marble quarries belonging to the factory, ratio of produced products sold on the internal market (%), ratio of produced products sold on the external market (%), monthly costs of laborer employed in the factory, monthly electricity costs of the factory, monthly water costs of the factory, monthly maintenance costs of the factory, monthly fuel costs of the factory, and average monthly socket costs of the factory. Whereas outputs are; monthly produced processed marble amount (m^2) and number of produced product varieties. They give some recommendations to the administrators of the factories for upgrading their production levels by summarizing the deficiencies of the factories, related with the results of the study. They mentioned that, by following these recommendations, the efficiencies of the factories will increase, and with increased and efficient productions, the importance of Afyonkarahisar city will be the highest in Turkey [10].

The study of Akgöbek, Ö. & Emre Yaku, E. (2014) aimed to examine the efficiency level of sectors operating in manufacturing industry in Turkey regarding the years between 1996-2008 via data envelopment analysis (DEA) and artificial neural network (ANN) to evaluate it from the financial aspect. This study is composed of two stages. Firstly, the efficiency of 14 sectors under manufacturing sector has been calculated by using DEA. Then, the efficiency scores of the sectors have been realized to be estimated by using the techniques of artificial neural networks. The input and output variables to be selected for the efficiency measurement of the sectors should have the best representative quality in estimating the efficiency. Considering this matter, inputs include: current ratio, total debt/equity capital, and tangible fixed assets / (long-term liabilities + equity capital) and the outputs are; stocks/current asset, net profit margin, active rate of return, and interest expense/net sales [11].

Baran et al. (2016) compared the technical efficiency of 12 sectors manufacturing basic metals and metal products in Poland. This study presented the use of data envelopment analysis models, to determine overall technical efficiency, pure technical efficiency and scale efficiency of metallurgical branches in Poland. The average technical efficiency of metallurgical industry in Poland was quite high. The analysis gave a possibility to create a ranking of sectors. Three branches were found to be fully efficient: manufacture of basic iron and steel and of ferroalloys, manufacture of basic precious and other non - ferrous metals and manufacture of tubes, pipes, hollow profiles and related fittings, of steel. The results pointed out the reasons of the inefficiency and provide improving directions for the inefficient sectors [12].

III. METHODOLOGY

A. Data Envelopment Analysis

Data envelopment analysis (DEA) is a mathematical programming approach developed in operations research and management science over the last decades for measuring the relative efficiency of a set of production systems, or decision making units (DMUs), with multiple inputs and multiple outputs. It originally was developed by Charnes, Cooper, and Rhodes (1978) under the assumption of constant returns to scale (CRS) and was extended by Banker, Charnes, and Cooper (1984) to include variable returns to scale (VRS) [13, 14].

- *The CCR Model*

The DEA model originally proposed by Charnes, Cooper, and Rhodes [13] is called the CCR model (which is named after the first letters of their names). In this model, the efficiency of any DMU is obtained as the maximum of a ratio of weighted outputs to weighted inputs subject to the condition that; the similar ratios for every DMU be less than or equal to unity. They assumed that there are n of DMU_S to be evaluated, where every DMU_j $(j = 1,2, \dots \dots, n)$ consumes varying amounts of m different inputs $x_{ij}(i = 1,2, \dots \dots, m)$ to produce s different output $y_{rj}(r = 1,2, \dots \dots, s)$. With decision variables outputs weights $u_r(r = 1,2, \dots, s)$ and inputs weights v_i $(i = 1,2, \dots, m)$ being selected, the mathematical formulation of the method is summarized as follows:

$$max \qquad h_0 = \frac{\sum_{r=1}^{s} u_r \; y_{r0}}{\sum_{i=1}^{m} v_i \; x_{i0}}$$

$$Subject\ to: \quad \frac{\sum_{r=1}^{s} u_r \; y_{rj}}{\sum_{i=1}^{m} v_i \; x_{ij}} \leq 1 \qquad ; \qquad j = 1,2, \dots \dots, n$$

$$u_r, v_i \geq 0 \qquad ; \quad r = 1,2, \dots \dots, s \quad ; \quad i = 1,2, \dots \dots, m$$

(1)

Hence, the fractional CCR model (1) evaluates the relative efficiencies of n decision making units (DMUs), each of them with m inputs and s outputs by maximizing the ratio of h_0.

- *The BCC Model*

Banker, Charnes, and Cooper 1984 [14] introduced the BCC model (which is named after the first letters of their names). This model is an extension of the CCR model. The primary difference between the two models (CCR and BCC) is the treatment of returns to scale. CCR model assumes constant returns to scale (CRS) while BCC model assumes variable returns to scale (VRS). The BCC ratio model differs from the CCR ratio model (1), by an additional variable as follows:

$$max \qquad h_0 = \frac{\sum_{r=1}^{s} u_r \, y_{r0} - c_0}{\sum_{i=1}^{m} v_i \, x_{i0}}$$

$$Subject\ to: \quad \frac{\sum_{r=1}^{s} u_r \, y_{rj} - c_0}{\sum_{i=1}^{m} v_i \, x_{ij}} \leq 1 \qquad ; \qquad j = 1,2, \ldots \ldots, n$$

$$u_r, v_i \geq 0 \quad ; \quad r = 1,2, \ldots \ldots, s \quad ; \quad i = 1,2, \ldots \ldots, m$$

$$c_0 \quad unrestricted\ in\ sign$$

(2)

Where c_0 is the new variable that separates scale efficiency from technical efficiency in the CCR model [15, 1].

B. Fuzzy Data Envelopment Analysis

Classical DEA models assume that input and output values should be certain (crisp data). However, in real-world problems inputs and outputs are often imprecise and vague. To deal with imprecise or vague data, fuzzy set theory has become an effective method. Recently, Fuzzy set theory has been applied to a wide range of fields such as management science, decision theory, artificial intelligence, computer science, expert systems, logic, control theory and statistics [16]. Sengupta (1992) was the first to introduce a fuzzy mathematical programming approach in which fuzziness was incorporated into DEA by allowing both the objective function and the constraints to be fuzzy. The author explored the use of fuzzy set theory in decision making. In his study, three types of fuzzy models (fuzzy mathematical programming, fuzzy regression and fuzzy entropy) were presented to illustrate the types of decisions and solutions that were achievable, when the data are vague and prior information is inexact and imprecise [17].

- *The Fuzzy CCR Model*

Assume that there are n of DMU_S to be evaluated, where every DMU_j $(j = 1,2, \dots \dots, n)$ consumes varying amounts of m different inputs $\tilde{x}_{ij}(i = 1,2, \dots \dots, m)$ to produce s different outputs $\tilde{y}_{rj}(r = 1,2, \dots \dots, s)$. Where $(\tilde{x}_{ij}, \tilde{y}_{rj})$ represent, respectively, the fuzzy input and fuzzy output of the jth DMU_j $(j = 1,2, \dots \dots, n)$. With decision variables outputs weights u_r $(r = 1,2, \dots \dots, s)$ and inputs weights v_i $(i = 1,2, \dots \dots, m)$ being selected, the fractional CCR model with fuzzy data can be formulated as follows:

$$\max \quad h_0 = \frac{\sum_{r=1}^{s} u_r \, \tilde{y}_{r0}}{\sum_{i=1}^{m} v_i \, \tilde{x}_{i0}}$$

$$\text{Subject to:} \quad \frac{\sum_{r=1}^{s} u_r \, \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \, \tilde{x}_{ij}} \leq 1 \quad ; \quad j = 1,2, \dots \dots, n$$

$$u_r, v_i \geq 0 \quad ; \quad r = 1,2, \dots \dots, s \quad ; \quad i = 1,2, \dots \dots, m$$

(3)

Where "~" indicate the fuzziness.

- *The Fuzzy BCC Model*

By the same way, the fractional BCC model with fuzzy data is given as follows:

$$\max \quad h_0 = \frac{\sum_{r=1}^{s} u_r \, \tilde{y}_{r0} - c_0}{\sum_{i=1}^{m} v_i \, \tilde{x}_{i0}}$$

$$\text{Subject to:} \quad \frac{\sum_{r=1}^{s} u_r \, \tilde{y}_{rj} - c_0}{\sum_{i=1}^{m} v_i \, \tilde{x}_{ij}} \leq 1 \quad ; \quad j = 1,2, \dots \dots, n$$

$$u_r, v_i \geq 0 \quad ; \quad r = 1,2, \dots \dots, s \; ; \quad i = 1,2, \dots \dots, m$$

$$c_0 \quad unrestricted\ in\ sign$$

(4)

Where "~" indicate the fuzziness.

The interpretation of constraints of FCCR and FBCC models is similar to the crisp CCR and BCC models. The difference between the two models resides on the manner of resolution. The crisp CCR model can be simply solved by a standard LP solver. For the FCCR model, the resolution is more difficult and requires methods for fuzzy sets [18].

- *The Interval DEA*

In this section, our attention will be focused on interval fuzzy numbers. In a condition that all inputs and outputs are not totally available due to uncertainties, these values are only known to lie within the upper and lower bounds represented by intervals $[{x_{ij}}^L, {x_{ij}}^U]$ and $[{y_{rj}}^L, {y_{rj}}^U]$, where ${x_{ij}}^L > 0$ and ${y_{rj}}^L > 0$. In order to deal with such an uncertain situation, the following pair of linear fractional models has been developed to generate the upper and lower bounds of interval efficiency for each DMU. Therefore, model (3) can be re-written as follows [19]:

$$max \qquad h_0^U = \frac{\sum_{r=1}^{s} u_r \, {y^U}_{r0}}{\sum_{i=1}^{m} v_i \, {x^L}_{i0}}$$

$$Subject\ to: \quad \frac{\sum_{r=1}^{s} u_r \, {y^U}_{rj}}{\sum_{i=1}^{m} v_i \, {x^L}_{ij}} \leq 1 \quad ; \quad j = 1,2, \dots \dots, n$$

$$u_r, v_i \geq 0 \quad ; \quad r = 1,2, \dots \dots, s \quad ; \quad i = 1,2, \dots \dots, m$$

(5)

$$max \qquad h_0^L = \frac{\sum_{r=1}^{s} u_r \, {y^L}_{r0}}{\sum_{i=1}^{m} v_i \, {x^U}_{i0}}$$

$$Subject\ to: \quad \frac{\sum_{r=1}^{s} u_r \, {y^U}_{rj}}{\sum_{i=1}^{m} v_i \, {x^L}_{ij}} \le 1 \qquad ; \qquad j = 1,2, \dots \dots, n$$

$$u_r, v_i \ge 0 \quad ; \quad r = 1,2, \dots \dots, s \quad ; \quad i = 1,2, \dots \dots, m \qquad (6)$$

Where h_0^U stands for the upper bound of the best possible relative efficiency of DMU_0, and h_0^L stands for the lower bound of the best possible relative efficiency of DMU_0.

Demir, E. (2014) [20] suggested solving the previous two models (5) and (6) by changing the crisp data into interval data. Upper and lower frontier data were calculated by adding and removing standard errors to each variable, and so each data was turned into interval data. To calculate upper frontier efficacy scores, upper frontier values of the output data and lower frontier values of the input data were used. When it came to the lower frontier efficacy scores, lower frontier values of the output data and upper frontier values of the input data were used. The formulas are:

Upper frontier data = Available data + Standard Error

Lower frontier data = Available data - Standard Error (7)

- *The suggested Methods*

Although Demir [20] suggested a method to change the crisp data into interval data by using standard errors of the variables to define the data as interval as mentioned before, this method sometimes gives negative values when the lower and upper bounds are being calculated depending on the nature of the data. The problem here is that, most software packages of DEA cannot treat with negative values. So, the method of Demir is being improved to eliminate this problem. In this study, three approaches are suggested to express crisp data as interval data in the form of lower and upper bounds. The first one is expressed as follows:

Lower bound data =original data - Standard Error*0.05

Upper bound data =original data + Standard Error *0.05 (8)

In the second approach, upper and lower frontier data are calculated by adding and removing ratio of standard deviations (one percent) to each variable as follows:

Lower bound data =original data - Standard deviation *0.01

Upper bound data =original data + Standard deviation *0.01 (9)

In the third approach, the ratio of standard deviations (one percent) is changed to five percent as follows:

Lower bound data =original data - Standard deviation *0.05

Upper bound data =original data + Standard deviation *0.05 (10)

To apply the suggested intervals; the data should be distributed as a normal distribution. In other words, these techniques assume that the variables are normally distributed. If a measurement variable does not fit a normal distribution, data transformations should be made. Data transformations such as square root, log, and inverse are commonly used tools that can serve many functions in quantitative analysis of data for improving the normality of variables.

IV. APPLICATION AND RESULTS

Manufacturing industries play a very significant role in whole economy. So, utilizing classical and interval DEA models, this paper examines the technical efficiency level of Ethiopian manufacturing industries for the year of 2008. The data used in this application was extracted from the Central Statistical Agency (CSA) of Ethiopia database. The data is taken from Hailu, K. B. and Tone, K. (2014) [21]. For the purpose of efficiency measurement, single-output and 3-input production technology for Ethiopian manufacturing is being used. Output is measured by the gross value of all outputs produced by the firm. The inputs include: (i) the number of employees measured by the sum of permanent and temporary workers, (ii) capital input measured by the net value of fixed assets at the end of the survey year, (iii) intermediate inputs aggregated as the sum of the values of raw materials, fuel and lubricating oil, electricity, wood and charcoal for energy for each establishment and other industrial costs. The types of industries that are used in the application include manufacture of: (1) Food and beverage, (2) Textiles, (3) Wearing apparel, (4) Tanning, leather and footwear, (5) Wood and wood products, (6) Paper and printing, (7) Chemical and chemical products, (8) Rubber and plastics products, (9) Non-metallic mineral products, (10) Fabricated metal products (11) Basic iron and steel, (12) Machinery and equipment, (13) Motor vehicles and (14) Furniture (see Appendix A).

To evaluate the relative efficiency values by using classical and interval DEA models, the steps are made as follows: first, calculating the efficiency values of classical DEA models (CCR / BCC). Second, testing whether measurement variables fit a normal distribution or not. Finally, calculating the efficiency values of interval DEA models (CCR / BCC) based on the suggested methods; formulas (8, 9, 10).

For solving data envelopment analysis (DEA) models, MaxDEA package has been employed. Efficiency values of classical DEA models (CCR / BCC) are calculated as shown in table (1):

Table (1) : Calculated efficiency values with classical DEA models (CCR / BCC)

DMU	Efficiency scores with CCR model	Efficiency scores with BCC model
S1	1	1
S2	0.6372	0.6999
S3	0.7139	0.72
S4	0.7769	0.7819
S5	1	1
S6	0.9501	1
S7	0.8573	0.8645
S8	0.7568	0.7846
S9	0.9945	1
S10	1	1
S11	0.9783	0.9878
S12	0.6974	1
S13	1	1
S14	0.7837	0.7862

In table (1), according to CCR model results; only four units are efficient and the rest of the units are deemed inefficient relative to other efficient units. While the BCC model is more flexible and allows more units to be efficient. So, seven units are efficient and the rest of the units are deemed inefficient relative to other efficient units.

To apply the suggested intervals (8, 9, 10); the data should be distributed as a normal distribution as mentioned before. This assumption was examined by SPSS program by using Kolmogorov-Smirnov test and it is found that all variables are normally distributed.

Efficiency values of interval DEA models (CCR / BCC) based on five percent of standard error, the formulas (8), are calculated and placed on table (2) for interval CCR model and also placed on table (3) for interval BCC model as follows:

Table (2) : Lower and upper frontier efficiency scores with interval CCR model

DMU	Lower efficiency values	Upper efficiency values
S1	1	1
S2	0.702	0.5672
S3	0.611	0.6263
S4	0.7817	0.7661
S5	0.9379	1
S6	1	0.8766
S7	0.8632	0.8436
S8	0.7449	0.7667
S9	0.9852	1
S10	1	1
S11	0.9566	0.9824
S12	0.5742	0.8386
S13	1	1
S14	0.7462	0.7405

In table (2), according to lower efficiency scores; only four units are efficient. While in the upper efficiency scores, five units are efficient and the rest of the units are deemed inefficient relative to other efficient units.

According to the results of interval BCC model as shown in table (3), seven units are efficient and the rest of the units are deemed inefficient relative to other efficient units in both lower and upper efficiency scores.

Table (3) : Lower and upper frontier efficiency scores with interval BCC model

DMU	Lower efficiency values	Upper efficiency values
S1	1	1
S2	0.7041	0.6955
S3	0.7401	0.6965
S4	0.7845	0.7792
S5	1	1
S6	1	1
S7	0.8665	0.8624
S8	0.7876	0.7816
S9	1	1
S10	1	1
S11	0.9882	0.9873
S12	1	1
S13	1	1
S14	0.7978	0.7732

Efficiency values of interval DEA models (CCR / BCC) based on one percent of standard deviation, the formulas (9), are calculated and placed on tables (4) and (5) for interval CCR and BCC models, respectively, as follows:

Table (4) : Lower and upper frontier efficiency scores with interval CCR model		
DMU	**Lower efficiency values**	**Upper efficiency values**
S1	1	1
S2	0.6939	0.5831
S3	0.6394	0.6449
S4	0.7828	0.7688
S5	1	1
S6	1	0.8938
S7	0.8647	0.847
S8	0.7486	0.7642
S9	0.9876	1
S10	1	1
S11	0.9681	0.9814
S12	0.6035	0.8018
S13	1	1
S14	0.7629	0.7505

Table (4) showed that only five units are efficient and the rest of the units are deemed inefficient relative to other efficient units according to both of lower and upper efficiency scores.

According to the results of interval BCC model as shown in table (5), seven units are efficient and the rest of the units are deemed inefficient relative to other efficient units in both lower and upper efficiency scores (i.e. the results are similar to table 3).

Table (5) : Lower and upper frontier efficiency scores with interval BCC model		
DMU	**Lower efficiency values**	**Upper efficiency values**
S1	1	1
S2	0.7031	0.6966
S3	0.7353	0.7028
S4	0.7839	0.7799
S5	1	1
S6	1	1
S7	0.866	0.863
S8	0.7869	0.7824
S9	1	1
S10	1	1
S11	0.9881	0.9874
S12	1	1
S13	1	1
S14	0.795	0.7766

Efficiency values of interval DEA models (CCR / BCC) based on five percent of standard deviation, the formulas (10), are calculated and placed on tables (6) and (7) for interval CCR and BCC models, respectively, as follows:

Table (6) : Lower and upper frontier efficiency scores with interval CCR model		
DMU	**Lower efficiency values**	**Upper efficiency values**
S1	1	0.7422
S2	0.6627	0.3724
S3	0.3536	0.4427
S4	0.7639	0.6351
S5	0.0758	1
S6	0.9625	0.6606

Table (6) : Lower and upper frontier efficiency scores with interval CCR model		
DMU	**Lower efficiency values**	**Upper efficiency values**
S7	0.8388	0.7101
S8	0.7037	0.6363
S9	0.9601	0.7707
S10	1	1
S11	0.8209	0.935
S12	0.3102	1
S13	0.8148	1
S14	0.5222	0.5675

Table (6) showed that, only two units are efficient according to lower efficiency scores. While in the upper efficiency scores, four units are efficient and the rest of the units are deemed inefficient relative to other efficient units.

The results of interval BCC model as shown in table (7) demonstrate that seven units are efficient according to lower efficiency scores; eight units are efficient according to upper efficiency scores and the rest of the units are deemed inefficient relative to other efficient units.

Table (7) : Lower and upper frontier efficiency scores with interval BCC model		
DMU	**Lower efficiency values**	**Upper efficiency values**
S1	1	1
S2	0.7152	0.6954
S3	0.7829	0.6058
S4	0.7915	0.7752
S5	1	1
S6	1	1
S7	0.8716	0.8609

Table (7) : Lower and upper frontier efficiency scores with interval BCC model		
DMU	**Lower efficiency values**	**Upper efficiency values**
S8	0.7954	0.7741
S9	1	1
S10	1	1
S11	0.9892	1
S12	1	1
S13	1	1
S14	0.824	0.7769

V. FINAL RESULTS AND CONCLUSION

The final results for the efficient units via classical and interval DEA models (the suggested approaches using five percent of standard error (S.E*0.05), one percent of standard deviation (S.D*0.01), and five percent of standard deviation (S.D*0.05) intervals) are summarized in the following tables (8) and (9) for the CCR and BCC models, respectively, as follows:

Table (8): Efficient units using CCR model						
Classical DEA	Lower and upper frontier efficient (S.E*0.05)		Lower and upper frontier efficient (S.D*0.01)		Lower and upper frontier efficient (S.D*0.05)	
	Lower	Upper	Lower	Upper	Lower	Upper
S1 S5 S10 S13	S1 S6 S10 S13	S1 S5 S9 S10 S13	S1 S5 S6 S10 S13	S1 S5 S9 S10 S13	S1 S10	S5 S10 S12 S13

Table (9): Efficient units using BCC model						
Classical DEA	Lower and upper frontier efficient (S.E*0.05)		Lower and upper frontier efficient (S.D*0.01)		Lower and upper frontier efficient (S.D*0.05)	
	Lower	Upper	Lower	Upper	Lower	Upper
S1 S5 S6 S9 S10 S12 S13	S1 S5 S6 S9 S10 S12 S13	S1 S5 S6 S9 S10 S12 S13	S1 S5 S6 S9 S10 S12 S13	S1 S5 S6 S9 S10 S12 S13	S1 S5 S6 S9 S10 S12 S13	S1 S5 S6 S9 S10 S11 S12 S13

Table (8) showed the efficient units using CCR model in different cases. According to these results, only unit 10 remains efficient in all cases and the suggested approaches that use five percent of standard error (S.E*0.05) and one percent of standard deviation (S.D*0.01) intervals are seemed to be better than the others.

According to the results of BCC model in table (9), there is no difference between efficient units in all cases and all the results are the same; so seven units, namely, S1, S5, S6, S9, S10, S12 and S13 were identified as the best practice units.

Another point of view, according to the results of both CCR and BCC models, Food and beverage, Wood, Paper and printing, Non-metals, Basic iron and steel, Machinery and equipment, and Motor vehicles industries are identified as the best performance units. So, we recommend that the decision makers should pay more attention to other inefficient industries to improve their efficiencies.

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Appendix (A)

(Original dataset for the year 2008)

DMUs	Industry	Companies	Labor (I)	Capital (I)	Intermediate (I)	Production (O)
S1	Food and beverage	٣٤٩	٣٥٣٩١.١٤٥	١٧٨٨١٦٨٢٢	١٤٩٥٢٣٢٢٤٦	٤٠٦٨٠٠٣٤٩٧
S2	Textiles	٢١	٨	١	٣٩٥١٨٨٣٧٧	٦٩٢١٨٧٧٧٤
S3	Wearing apparel	٢٦	١٨٣٦٣.٧٩١	٣٨١٠٥٧٤٦٩	٦٥٣١٨٠٣٢.١	١٢٩٧٥٠٨٢٥
S4	Tanning, leather	٦٢	٦	١٠٤٣٥٣٧٣٢	٥٦١٠٦٤٩٦٨	٨٩٠٦٤٧٩٦٧
S5	&footwear	٢٨	٤٨١٥.٧٩١٦	٣٦٣٣٦١٧٤٧	٢٠١٧٦٦٠٠.١	٥٦٧٢٩٦٦٠.١
S6	wood	٩٦	٥	٦٨٧١١٤١.٣	٣٢٠١٥٧١٦٤	٦٣٨٥٥٨٠١٧
S7	Paper and printing	٦١	٧٨٨٧.٥٦٢٥	١	٤٨٢٢٣١٥٢٢	٨١٨٧٨٠٥٩٨
S8	Chemicals	٧٥	١٩٥٩.٨١٢٤	١٨٠٥٥٠٠١٨	٤١٧٦٥٧٨٢٩	٧٢٧٠٧٨٦٧٩
S9	Rubber and plastics	٢٤٣	٨	٢٨٦٨٣٤١٦١	٥٧٦١٨٣١٥١	١٣٧٧٣٩٢٦٩٦
S10	Non-metals	١٤	٧٩٦٧.٦٤٥٨	٣٦١٣٨٥٦٧٢	٦٣٣٣٤٠١٦٤	١٠٠٢٦٠٩٤٠٨
S11	Basic iron and steel	٤٩	٥	٦٧٧٣٥٦٦٦٧	١٨٩٥٤٩٩٩١	٣٤٢٦١٥٧٦٦
S12	Fabricated metals	٩	٦٤٢٦.٥٠٠٠	٢١٢٢٢٣٠٠١	٦٧٠٧٥٠٢٨.٣	١١٦٤٢٩٨٧٤
S13	Machinery and equipment	١٠	١	٨٨٣٠٩٦٤٦.	٢٣٣٤٣١٧٨٦	٣٢٧٠٣١٨٥٠
S14	Motor vehicles	170	٧٠٨٣.١٦٦٦	٧	87966624.3	189933023
	Furniture		٥	٩٢٠١٨٥٣٦.		
			١٠٣٠٠.٧٥	٢		
			١٨٢١.٦٠٤١	٦٠٩٦٨٧٦٥.		
			٨	٨		
			٢٤٨٢.٨٩٥٨	93204031.8		
			٤			
			١٣٠٣.٥			
			١٢١٤.٤١٦٦			
			٦			
			4529.18749			

Stress-Strength Reliability of (Y<X) for Inverse Wiebull distribution using Hybrid Censored Samples

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Stress–Strength Reliability of (Y<X) for Inverse Wiebull distribution using Hybrid Censored Samples

Abstract

In this article, we introduce the estimation of stress–strength reliability $R=P(Y<X)$, when X and Y are two independent inverse Weibull (IW) lifetime models having the same shape parameters but different scale parameters under hybrid censored samples. First, the maximum likelihood estimator and its asymptotic distribution are obtained. Based on the asymptotic distribution, the confidence interval of R can also be obtained and finally the Bayesian estimate of R using squared loss function is proposed.

Key Words: Inverse Weibull distribution; Hybrid censoring; Bayesian estimation; squared loss function.

Introduction

Weibull distribution was introduced by Weibull in 1935, from this time it was used very effectively in analyzing various lifetime data. The hazard function of Weibull is decreasing or increasing depending on the shape parameter. When the data has a non-monotone hazard function, the Weibull distribution cannot be used, therefore, if the empirical study indicates that the hazard function of the underlying distribution is not monotone, and it is unimodal, then inverse Weibull (IW) distribution may be used to analyze such data set (Kundu and Howlader (2010)). It is a lifetime probability distribution that can be used in the reliability engineering discipline. The inverse Weibull distribution has the ability to model failure rates, which is quite common in reliability and biological studies. The IW distribution plays some important roles in other areas, such as describing the degradation phenomena of mechanical components, describing the context of a load strength relationship for a component and providing the good fit to survival data. Extensive work has been done on the IW distribution, such as Keller and Kamath (1982), Calabria and Pulcini (1989, 1990, 1992, 1994), Jiang and Xiao (2003) Mahmoud, Sultan and

The inference of stress–strength reliability in statistics is an important topic of interest. It has many applications in practical areas. Let X be the strength of a component and Y be the stress applied to the component, then R can be considered as a measure of the component performance. The component fails if and only if at any time the applied stress is greater than its strength (Li and Hao 2017). The estimation of the stress–strength parameter R has attracted much attention recently in the statistical literature authors. Singh et. al (2015) present estimate the stress–strength reliability parameter $R = P(Y < X)$, when X and Y are independent inverted exponential random variable. They discussed the maximum likelihood and Bayesian estimator R and its asymptotic distribution are obtained. Bi and Gui (2017) present estimating stress–strength reliability for inverse Weibull using complete data when X and Y are independent but not identically inverse Wiebull (IW) distributed random variables. They used an approximate maximum likelihood estimator. The asymptotic confidence interval and two bootstrap intervals are obtained. Using the Gibbs sampling technique, Bayesian estimator and the corresponding credible interval are obtained. And Analysis of a real dataset is performed.

Azimi et al (2012) presented the Bayesian estimators of parameter and Reliability function are obtained of Rayleigh distribution using asymmetric loss functions such, LINEX loss function, Precautionary loss function, entropy loss function under a progressively type II censored sample. Li and Hao (2017) introduced The maximum likelihood estimator and the Bayesian estimate of R and the corresponding confidence intervals when X and Y are independent identically inverse Wiebull (IW) distributed random variables for IW using complete data.

For more details see, for example Awad *et al.* (1981), Kundu and Gupta (2005, 2006), Kundu and Ragab (2009) Asgharzadeh *et al.* (2013) for complete samples. Recently, some authors have also investigated the estimation of of R for some lifetime distributions based on record and censored data. See for example, the work of Baklizi (2008), Asgharzadeh *et al.* (2015),

Type-I and Type-II censoring schemes are the two most popular censoring schemes which have been used in practice. In Type-I censoring scheme, the experimental time is fixed, but the number of failures is random, whereas in Type-II censoring scheme, the experimental time is random but the number of failures is fixed. A hybrid censoring is a mixture of Type–I and Type–II censoring schemes and it can be described as follows. Suppose n identical units are put on a life test. The lifetimes of the sample units are independent and identically distributed (i.i.d) random variables. The test is terminated when a pre–specified number r out of n units has failed or a pre–determined time T, has been reached. It is also assumed that the failed items are not replaced. Therefore, in hybrid censoring scheme, the experimental time and the number of failures will not exceed T and r, respectively Childs et al(2003). Under the hybrid censoring scheme we have one of the two following types of observations:

1: $(x_{1:n}<x_{2:n}<\ldots<x_{h:n})$ if $x_{h:n}<T$

2: $(x_{1:n}<x_{2:n}<\ldots<x_{k:n})$ if $k<h$, $x_{k:n}<T<x_{k+1:n}$

Where $(x_{1:n}<x_{2:n}<\ldots)$ are the observed ordered failure time to the experimental units.

Asgharzadeh et.al (2015).

Inverse Weibull Distribution

The probability density function of the Weibull distribution is given by:

$$f(x;\alpha,\beta)=\alpha\beta\, z^{\alpha-1}e^{-\beta z^{\alpha}} \quad z,\alpha,\beta>0 \qquad (1)$$

Where α is the shape parameter and β is the scale parameter

Let X denote the random variable from Weibull model in equation (1), let $X=\frac{1}{Z}$

Then the random variable X is said to have a two-parameter IW distribution and its probability density function (PDF) is given by

$$f(x;\alpha,\beta)=\alpha\beta\, x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}, \quad x,\alpha,\beta>0\,. \qquad (2)$$

And the cumulative distribution function (CDF), reliability function and hazard function are given by:

$$F(x)=e^{-\beta x^{-\alpha}}, \quad x,\beta>0 \qquad (3)$$

$$R(x)=1-e^{-\beta x^{-\alpha}} \quad x,\beta>0 \qquad (4)$$

$$h(x)=\frac{f(x)}{R(x)}=\frac{\alpha\beta\, x^{-(\alpha+1)}e^{-\beta\, x^{-\alpha}}}{1-e^{-\beta\, x^{-\alpha}}} \qquad (5)$$

Therefore the stress–strength structural R= P(Y<X) is reliability parameter R.

Thus R= P(Y<X) is the characteristic of the distribution of X and Y.

Then reliability of the component is:

$$R=\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x)\,f(y)\;\;dy\,dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{x} \alpha\beta_1\, x^{-(\alpha+1)}e^{-\beta_1 x^{-\alpha}}\;\alpha\beta_2\, y^{-(\alpha+1)}e^{-\beta_2 y^{-\alpha}}\;\;dy\,dx$$

$$\therefore R=\frac{\beta_2}{\beta_1+\beta_2} \qquad (6)$$

Maximum Likelihood Estimation of R

To derive the MLE of R, based on hybrid censored data on both variables first we obtain the MLE's of α,β_1 and β_2, let X=(X_1,X_2,…,X_{r1}) for =($X_{1:n}$,$X_{2:n}$,…,$X_{r1:n}$) is a hybrid censored sample from exp (α,β_1) with censored scheme (r_1,T_1) and Y=(Y_1,Y_2,…,Y_{r2}) for =($Y_{1:n}$,$Y_{2:n}$,…,$Y_{r1:n}$) is a hybrid censored sample from exp (α,β_2) with censored scheme (r_2,T_2). Therefore, the likelihood function of α,β_1 and β_2 is given by:

$$L(\alpha,\beta_1,\beta_2)=c_1(\prod_{i=1}^{r_1} f(x)(1-F(v_1))^{n-r_1})c_2(\prod_{j=1}^{r2} f(y)(1-F(v_2))^{m-r_2}) \tag{7}$$

$c_1 = n(n-1)(n-2)\ \ldots\ (n-r_1+1)$; $c_2 = m(m-1)(m-2)\ \ldots\ (m-r_2+1)$

$V_1 = min(X_{R1}\ ;\ T_1)$; $V_2 = min(X_{R2}\ ;\ T_2)$;

Using (1) and (2) without the multiplicative constant, the likelihood function of α,β_1 and β_2 can be writing as:

$$L(data\,|\,\alpha,\beta_1,\beta_2)=\alpha^{r_1+r_2}\beta_1^{r_1}\beta_2^{r_2}\prod_{i=1}^{r_1}x_i^{-\alpha-1}\prod_{j=1}^{r_2}y_j^{-\alpha-1}$$
$$*\exp[-\beta_1(\sum_{i=1}^{r_1}x_i^{-\alpha}+(n-r_1)v_1^{-\alpha})-\beta_2(\sum_{j=1}^{r_2}y_j^{-\alpha}+(m-r_2)v_2^{-\alpha})] \tag{8}$$

Hence, the log likelihood function $L(data\,|\,\alpha,\beta_1,\beta_2)=l(\alpha,\beta_1,\beta_2)$ becomes

$$l(\alpha,\beta_1,\beta_2)=(r_1+r_2)\log\alpha+r_1\log\beta_1+r_2\log\beta_2-(\alpha+1)\sum_{i=1}^{r_1}\log x_i-(\alpha+1)\sum_{j=1}^{r_2}\log y_j$$
$$-\beta_1(\sum_{i=1}^{r_1}x_i^{-\alpha}+(n-r_1)v_1^{-\alpha})-\beta_2(\sum_{j=1}^{r_2}y_j^{-\alpha}+(m-r_2)v_2^{-\alpha}) \tag{9}$$

Differentiating the log-likelihood function $l(\alpha, \beta_1, \beta_2)$ partially with respect to α, β_1 and β_1 then equating to zero we have

$$\frac{\partial l}{\partial \alpha} = \frac{r_1 + r_2}{\alpha} - \sum_{i=1}^{r_1} \log x_i - \sum_{j=1}^{r_2} \log y_j - \beta_1 (\sum_{i=1}^{r_1} x_i^{-\alpha} \log x_i + (n - r_1) v_1^{-\alpha} \log v_1) - \beta_2 (\sum_{j=1}^{r_2} y_j^{-\alpha} \log y_j + (m - r_2) v_2^{-\alpha} \log v_2) = 0 \tag{10}$$

$$\frac{\partial l}{\partial \beta_1} = \frac{r_1}{\beta_1} - (\sum_{i=1}^{r_1} x_i^{-\alpha} + (n - r_1) v_1^{-\alpha}) = 0 \tag{11}$$

$$\frac{\partial l}{\partial \beta_2} = \frac{r_2}{\beta_2} - (\sum_{j=1}^{r_2} y_j^{-\alpha} + (m - r_2) v_2^{-\alpha}) = 0 \tag{12}$$

From (11) and (12), we can get

$$\hat{\beta}_1 = \frac{r_1}{\sum_{i=1}^{r_1} x_i^{-\alpha} + (n - r_1) v_1^{-\alpha}} \tag{13}$$

And

$$\hat{\beta}_2 = \frac{r_2}{\sum_{j=1}^{r_2} y_j^{-\alpha} + (m - r_2) v_2^{-\alpha}} \tag{14}$$

Then $\hat{\alpha}$ can be obtained as a solution of the non-linear equation as follow:

$$g(\alpha)=\frac{r_1+r_2}{\alpha}-\sum_{i=1}^{r_1}\log x_i-\sum_{j=1}^{r_2}\log y_j-\hat{\beta}_1(\sum_{i=1}^{r_1}x_i^{-\alpha}\log x_i+(n-r_1)v_1^{-\alpha}\log v_1)$$
$$-\hat{\beta}_2(\sum_{j=1}^{r_2}y_j^{-\alpha}\log y_j+(m-r_2)v_2^{-\alpha}\log v_2)=0$$

Here, $\hat{\alpha}$ can be obtained from the non−linear equation $h(\alpha)=\alpha$

where

$$h(\alpha)=\frac{(r_1+r_2)}{\Theta} \quad where$$
$$\Theta=\sum_{i=1}^{r_1}\log x_i-\sum_{j=1}^{r_2}\log y_j-\hat{\beta}_1(\sum_{i=1}^{r_1}x_i^{-\alpha}\log x_i+(n-r_1)v_1^{-\alpha}\log v_1)$$
$$+\hat{\beta}_2(\sum_{j=1}^{r_2}y_j^{-\alpha}\log y_j+(m-r_2)v_2^{-\alpha}\log v_2)$$
(15)

The MLE of stress−strength reliability R becomes

$$\therefore\hat{R}=\frac{\hat{\beta}_2}{\hat{\beta}_1+\hat{\beta}_2} \qquad (16)$$

$$\hat{R}=\frac{\dfrac{r_2}{\sum_{j=1}^{r_2}y_i^{-\hat{\alpha}}+(m-r_2)v_2^{-\hat{\alpha}}}}{\dfrac{r_1}{\sum_{i=1}^{r_1}x_i^{-\hat{\alpha}}+(n-r_1)v_1^{-\hat{\alpha}}}+\dfrac{r_2}{\sum_{j=1}^{r_2}y_j^{-\hat{\alpha}}+(m-r_2)v_2^{-\hat{\alpha}}}} \qquad (17)$$

Asymptotic Confidence Intervals of *R*

In this section asymptotic distribution of $\hat{\theta}=(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)$ and $\hat{R}$ firstly, then the asymptotic confidence interval of *R* can be obtained. Let us denote the expected Fisher information matrix of $\theta=(\alpha,\beta_1,\beta_2)$ as $I(\theta)=(I_{ij}(\theta); i, j=1,2,3)$ Therefore

$$I(\theta)=\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}=-E\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta_1} & \frac{\partial^2 l}{\partial \alpha \partial \beta_2} \\ \frac{\partial^2 l}{\partial \beta_1 \partial \alpha} & \frac{\partial^2 l}{\partial \beta_1^2} & \frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} \\ \frac{\partial^2 l}{\partial \beta_2 \partial \alpha} & \frac{\partial^2 l}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 l}{\partial \beta_2^2} \end{pmatrix} \tag{18}$$

Then we get that

$$I_{11}=\frac{r_1+r_2}{\alpha^2}+\beta_1\left(\sum_{i=1}^{r_1} x_i^{-\alpha}(\log x_i)^2+(n-r_1)v_1^{-\alpha}(\log v_1)^2\right)$$
$$+\beta_2\left(\sum_{j=1}^{r_2} y_j^{-\alpha}(\log y_j)^2+(m-r_2)v_2^{-\alpha}(\log v_2)^2\right) \tag{19}$$

$$I_{22} = \frac{r_1}{\beta_1^2}, \quad I_{33} = \frac{r_2}{\beta_2^2} \tag{20}$$

$$I_{12} = I_{21} = \sum_{i=1}^{r_1} x_i^{-\alpha} \log x_i + (n - r_1) v_1^{-\alpha} \log v_1 \tag{21}$$

$$I_{13} = I_{31} = \sum_{j=1}^{r_2} y_j^{-\alpha} \log y_j + (m - r_2) v_2^{-\alpha} \log v_2 \tag{22}$$

$$I_{23} = I_{32} = 0 \tag{23}$$

Let $A = I(\theta)^{-1}$ is the asymptotic variance– covariance matrix, where $I(\theta)^{-1}$ is inverting of Fisher information matrix. Therefore

$$A = \frac{1}{\nabla}\begin{pmatrix} I_{33}I_{22} & -I_{21}I_{33} & -I_{31}I_{22} \\ -I_{12}I_{33} & I_{31}I_{31} & -I_{12}I_{31} \\ -I_{13}I_{22} & I_{13}I_{21} & -I_{12}I_{21} \end{pmatrix}, \; and \; \nabla = -[I_{12}I_{21}I_{33} + I_{13}I_{22}I_{31}] \tag{24}$$

As n⟶∞,m⟶∞,

$$\sqrt{n}(\hat{\beta}_1 - \beta_1), \sqrt{n}(\hat{\beta}_2 - \beta_2), \sqrt{n}(\hat{\alpha} - \alpha) \to N(0, A^{-1}(\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)$$

To obtain asymptotic confidence interval for R, we proceed as follows (Rao, 1973) :

$$d_1(\beta_1,\beta_2)=\frac{\partial R}{\partial \beta_1}=\frac{-\hat{\beta}_2}{(\hat{\beta}_1+\hat{\beta}_2)^2}$$

$$d_2(\beta_1,\beta_2)=\frac{\partial R}{\partial \beta_2}=\frac{\hat{\beta}_1}{(\hat{\beta}_1+\hat{\beta}_2)^2}$$

This gives

$$\operatorname{var}(\hat{R})=\operatorname{var}(\hat{\beta}_1)d_1^2(\beta_1,\beta_2)+\operatorname{var}(\hat{\beta}_2)d_2^2(\beta_1,\beta_2)$$

$$\therefore \operatorname{var}(\hat{R})=I_{22}^{-1}\frac{2\hat{\beta}_2(\hat{\beta}_1+\hat{\beta}_2)}{(\hat{\beta}_1+\hat{\beta}_2)^4}+I_{33}^{-1}\frac{-2\hat{\beta}_1(\hat{\beta}_1+\hat{\beta}_2)}{(\hat{\beta}_1+\hat{\beta}_2)^4} \tag{25}$$

Then the asymptotic $100(1-\lambda\ )\%$ confidence interval for R would be (L,U).

Where

$$U=\hat{R}+z_{1-\frac{\lambda}{2}}\sqrt{\operatorname{var}(\hat{R})},$$

$$L=\hat{R}-z_{1-\frac{\lambda}{2}}\sqrt{\operatorname{var}(\hat{R})}$$

Where $z_{1-\frac{\lambda}{2}}$ is the $(1-\frac{\lambda}{2})^{th}$ percentile of the standard normal distribution and

$\hat{R}$ is given by equation (17)

Bayesian Estimation of R

In this section the Bayes estimation of *R* under the squared error loss can be obtained if the shape parameter α is known (non–informative distribution), the scale parameter $\beta_1 and \beta_2$ has a conjugate prior, which is a gamma prior. And consider the priors α, β_1 and β_2 are independent.

$$\pi_1(\beta_1) = \frac{a_1^{b_1}}{\Gamma(b_1)} \beta_1^{b_1 - 1} \exp(-a_1 b_1), \quad \beta_1 > 0$$

$$\pi_2(\beta_2) = \frac{a_2^{b_2}}{\Gamma(b_2)} \beta_2^{b_2 - 1} \exp(-a_2 b_2), \quad \beta_2 > 0$$

$$\pi_3(\alpha) = \frac{1}{\alpha}, \qquad \alpha > 0$$

Therefore, we can get the joint prior distribution α, β_1 and β_2 as following:

$$\pi(\alpha, \beta_1, \beta_2) = \frac{a_1^{b_1} a_2^{b_2}}{\alpha \Gamma b_1 \Gamma b_2} \beta_1^{b_1 - 1} \beta_2^{b_2 - 1} \exp -(a_1 \beta_1 + a_2 \beta_2),\ a_i, b_i > 0 \{i = 1,2\} \qquad (26)$$

Based on data the joint posterior density function of α, β_1 and β_2 can be obtained as following:

$$\pi(\alpha,\beta_1,\beta_2|data)=\frac{\pi(\alpha,\beta_1,\beta_2)L(data|\alpha,\beta_1,\beta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\beta_1,\beta_2)L(data|\alpha,\beta_1,\beta_2)\partial\alpha\partial\beta_1\partial_2} \quad (27)$$

Where $L(data|\alpha,\beta_1,\beta_2)$ is defined in (8)

Under squared-error loss function, the Bayesian estimator of α,β_1 and β_2 is the mean of the posterior density given by:

$$\hat{\beta}_{1s}=E(\beta_1|data)=\frac{1}{K}\int_0^\infty\int_0^\infty\int_0^\infty \beta_1\pi(\alpha,\beta_1,\beta_2)L(data|\alpha,\beta_1,\beta_2)\,\partial\alpha\partial\beta_2\partial\beta_1$$

$$\hat{\beta}_{2s}=E(\beta_2|data)=\frac{1}{K}\int_0^\infty\int_0^\infty\int_0^\infty \beta_2\pi(\alpha,\beta_1,\beta_2)L(data|\alpha,\beta_1,\beta_2)\,\partial\alpha\partial\beta_1\partial\beta_2$$

$$\hat{\alpha}_s=E(\alpha|data)=\frac{1}{K}\int_0^\infty\int_0^\infty\int_0^\infty \alpha\pi(\alpha,\beta_1,\beta_2)L(data|\alpha,\beta_1,\beta_2)\,\partial\beta_1\partial\beta_2\partial\alpha \quad (28)$$

Where

$$K=\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\beta_1,\beta_2)L(data|\alpha,\beta_1,\beta_2)\partial\alpha\,\partial\beta_1\partial\beta_2 \quad (29)$$

Therefore

$$\hat{R}=\frac{\hat{\beta}_{2s}}{\hat{\beta}_{1s}+\hat{\beta}_{2s}} \quad (30)$$

Simulation Study

In this section, we compare the performance of the MLE and Bayesian estimations (under the squared error loss function) of R under the different sample sizes and different parameter values by using Monte Carlo simulation. We compare the performances of the MLE and the Bayes estimates in terms mean squares errors (MSE) and initial estimate to $\alpha=1$.

Table (1): MSE for MLE and Bayesian estimation for R when m = n =40 and $(\alpha,\beta_1,\beta_2)=(1,1.5,1)$

r_1,T_1	r_2,T_2	MSE (MLE)	MSE(Bayesian)
(15,1)	(15,1)	0.0091	0.0083
	(25,1)	0.0087	0.0071
	(40,2)	0.0077	0.0069
(25,1)	(15,1)	0.0088	0.0077
	(25,1)	0.0079	0.0068
	(40,2)	0.0070	0.0061
(35,2)	(15,2)	0.0081	0.0071
	(25,2)	0.0073	0.0063
	(40,2)	0.0064	0.0054

Conclusion

In this paper the maximum likelihood estimation and the Bayesian estimation of *Pr(Y<X)* when X and Y are independent random variable which Continued Inverse Weibull distribution with the same shape parameter and different scale parameter is presented, It is assumed that the data are hybrid censored for both X and Y. and the asymptotic distribution and confidence intervals, is presented. We observed that the MLE and Bayesian estimations of R cannot be obtained in explicit form. So we used a Monte Carlo simulation which proved that the performances of the Bayes estimators are very satisfactory.

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